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Event-driven boundary control for time fractional diffusion systems under time-varying input disturbance

Fudong Ge[1,*] and YangQuan Chen[2]

Abstract— **This paper is concerned with the event-driven boundary state feedback control problem for the subdiffusion processes governed by time fractional diffusion systems with unknown time-varying input disturbance. To evaluate the unknown disturbance, we propose an estimator through filtering, which only requires that the Laplace transformation of the disturbance signal exists and is finite. Moreover, we study the stability of the closed-loop system using a state feedback event-driven control strategic via Backstepping technique. A positive lower bounded minimum inter-event time of the event-driven strategic is then presented to avoid the occurrence of Zeno phenomenon. Finally, we work out a numerical example to test the proposed method.**

Index Terms— **Time fractional diffusion systems; Event-driven control; Time-varying input disturbance; Backstepping.**

I. INTRODUCTION

Recently the studies of subdiffusion transport dynamics in complex systems have attracted a great deal of attention [1], [2], [3] and time fractional diffusion systems have been proved to be valuable tools to model them [4]. This is due to the fact that fractional derivative is defined as a kind of convolution and good at characterizing the subdiffusion processes. Note that the boundary feedback control for conventional parabolic distributed parameter systems (DPSs) has been widely studied in control communities. Realize that time fractional diffusion systems can be regarded as an extension of conventional parabolic DPSs, where the first order time derivative is generalized to a fractional derivative of order $\alpha \in (0,1)$. Together with the fact that disturbances can not be negligible for all practical applications, research on the boundary feedback control problem of time fractional diffusion systems with time-varying input disturbance should be interesting and challenging.

Motivated by the above considerations, in this paper, we investigate the boundary control problem of the following time fractional diffusion systems with a Caputo fractional derivative ${}_0^C D_t^\alpha$ of order $\alpha \in (0,1)$

$$\begin{cases} {}_0^C D_t^\alpha y(x,t) = \triangle y(x,t) + \sigma(x) y(x,t) \\ \qquad\qquad \text{in } (0,1)\times[0,\infty), \\ y(0,t)=0,\ y(1,t)=u(t)+d(t) \text{ in } [0,\infty), \\ y(x,0)=y_0(x) \text{ in } (0,1), \end{cases} \tag{1}$$

where $\triangle := \partial^2/\partial x^2$ is the Laplace operator, $\sigma \in C^1(0,1)$ and $y_0 \in L^2(0,1)$. Here $L^2(0,1)$ represents the usual square integrable function space endowed with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. Moreover, $u(t)$ is the control input which is to be designed to stabilize the system and $d(t)$ denotes the time-varying boundary input disturbance.

The applications of system (1) are rich in real world. As cited in [5], it is usually used to describe the reaction-diffusion processes in a spatially inhomogeneous environment. For example, the chemical reaction processes in dispersive transport media [6], reheating processes of the heterogeneous metal slabs [7], or the flow through porous media with a source [8]. Then here (1) can be viewed as a model of a thin rod with not only the heat loss on the right-sided $(x=1)$ but also the heat generation inside the rod in a spatially inhomogeneous environment.

To deal with the disturbances, several methods have been widely used for the conventional parabolic DPSs. A sliding mode control strategic has been applied in [9] to stabilize heat equations with boundary disturbances, which shows a good robust performance. In [10], the authors designed a combined backstepping and sliding mode controller for one-dimensional unstable heat equation with boundary uncertainties. To investigate the quasilinear parabolic DPSs with time-varying uncertain variables, an extended Kalman filter based controller and a Galerkin's finite dimensional approximation method have been given in [11] and in [12] respectively. However, there is a need for further studies on the sliding mode control theory of time fractional diffusion systems. New approaches to estimate the boundary disturbances of system (1) are needed. Note that another method called uncertainty and disturbance estimator (UDE) based control has received much attention in the past two decades as shown in [13], [14], which has been extended to discuss the parabolic DPSs [15]. The UDE-based method only requires that the Laplace transformation of the disturbance signal exists and is finite. Then we shall adopt the UDE to estimate the time-varying input disturbance in system (1).

In addition, to reduce the workload of network and so as to save resources, an implementation where the transmission instants are defined based on a state-dependent criterion has

* Corresponding author.

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[1]School of Computer Science, China University of Geosciences, Wuhan 430074, PR China. He is also with the Hubei Key Laboratory of Intelligent Geo-Information Processing, China University of Geosciences, Wuhan 430074, PR China (Email: gefd@cug.edu.cn)

[2]Mechatronics, Embedded Systems and Automation Lab, University of California, Merced, CA 95343, USA (Email: ychen53@ucmerced.edu)

been introduced as an alternative [16], [17]. This is the event-driven control, where the control action is done only when the designed driven event is breached. Compared with the time-driven control scheme, event-driven control provides a natural way to efficiently reduce the number of control updates, which could lead to a lower (average) workload of network while guaranteeing the desired performance [18]. Besides, an event-driven scheme can save computation resources, energy resources for wireless communication via battery-powered devices and limited network resources. With these advantages, an idea of event-driven control has been widely performed to deal with several control and filtering issues finite-dimensional systems governed by ordinary differential equations [16], [17], [19].

The contribution of this paper is to consider the event-driven boundary feedback control for the infinite-dimensional time fractional diffusion system (1). To the best of our knowledge, no result is available on this topic. To realize this, it is supposed that the full-states of system (1) are available. Then we design an event-driven boundary state feedback controller based on the measured states via backstepping technique to asymptotically stabilize the closed-loop system. Besides, for the event-driven implementation, it is important to guarantee the existence of a positive lower bounded minimum inter-event time. Not only is this requirement significant to obtain our stabilization results, it also avoids the occurrence of Zeno phenomenon, i.e., to prevent the existence of an infinite number of transmissions in a finite time [20]. Moreover, we realize that in many practical cases, the availability of full-state measurements may be impossible due to the difficulties in measuring. To solve this limitation, some new methods should be introduced to estimate the whole states of studied systems. While interesting, we shall discuss these problems in our forthcoming work.

This paper is proceed as follows. Some basic results to be used thereafter are recalled in the next section. In Section 3, our main results on the design and implementation of event-driven controller are presented using backstepping technique. A numerical example is finally included.

II. Preliminary results

We first recall some basic results to be used thereafter.

Definition 1: [21] The Riemann-Liouville fractional integral of order $\alpha > 0$ for $y(\cdot,t) : [0,\infty) \to \mathbb{R}$ on t is given by

$$ {}_0I_t^{\alpha} y(\cdot,t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(\cdot,s)ds \tag{2}$$

provided that the right side is pointwise defined on $[0,\infty)$. Here $\Gamma(\alpha) := \int_0^\infty \tau^{\alpha-1}e^{-\tau}d\tau$ denotes the Gamma function.

Definition 2: [21] The Caputo fractional derivative of order $\alpha \in (0,1)$ for $y(\cdot,t) : [0,\infty) \to \mathbb{R}$ on t is defined as

$$ {}_0^C D_t^{\alpha} y(\cdot,t) = \begin{cases} \frac{\partial}{\partial t} y(\cdot,t), & \alpha = 1, \\ {}_0I_t^{1-\alpha} \frac{\partial y}{\partial t}(\cdot,t), & 0 < \alpha < 1 \end{cases} \tag{3}$$

provided that the right side is pointwise defined on $[0,\infty)$.

Consider the Laplace operator $\triangle$ with Dirichlet boundary conditions $y(0) = y(1) = 0$, let $\lambda_n = -n^2\pi^2$ and $\xi_n(x) = \sqrt{2}\sin(n\pi x)$. We obtain that

i) λ_n is the eigenvalue of $\triangle$;

ii) $\xi_n(x)$ is the eigenfunction of $\triangle$ corresponding to λ_n.

By [22], system (1) can not be stable if some eigenvalues of $\triangle + \sigma(x)$ are bigger than zero. Then we conclude that the system (1) is unstable if σ is positive and large enough.

To discuss the stabilization problem of system (1), we introduce the following special function

$$ E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)},\ \mathbb{R}(\alpha) > 0,\ t \in \mathbb{C}, \tag{4}$$

which is known as the Mittag-Leffler function in two parameter and in particular, we write $E_{\alpha,1}(t) = E_\alpha(t)$ for short when $\beta = 1$. If $\alpha = \beta = 1$, it reduces to the conventional exponential function. Then Mittag-Leffler function is usually regarded as an extension of the exponential function.

Next, we present some useful lemmas, which play a key role to obtain our results.

Lemma 1: [23] Let $\alpha < 2$, β be an arbitrary real number and $\frac{\pi\alpha}{2} < \theta < \min\{\pi, \pi\alpha\}$. If $\theta \leqslant |\arg(z)| \leqslant \pi$, $|z| \geqslant 0$, then

$$ \left|E_{\alpha,\beta}(z)\right| \leqslant \frac{M}{1+|z|} \tag{5}$$

holds for some constants $M > 0$.

By Lemma 1, if $\alpha \in (0,1)$, $\beta = 1$, $t \in \mathbb{R}$ and $t \geqslant 0$, we have $|\arg(-t)| = \pi \in [\theta,\pi]$ and

$$ |E_\alpha(-t)| \leqslant \frac{M}{1+t} \leqslant Mt^{-1},\ \alpha \in (0,1),\ t \geqslant 0. \tag{6}$$

This shows that Mittag-Leffler stability could imply asymptotical stability.

Lemma 2: Given $\varepsilon > 0$, $t \geqslant 0$, $\alpha \in (0,1)$, it follows that

$$ \int_0^t \frac{E_{\alpha,\alpha}(-\varepsilon(t-s)^\alpha)}{(t-s)^{1-\alpha}} s^{-\alpha} ds = \Gamma(1-\alpha)E_\alpha(-\varepsilon t^\alpha). \tag{7}$$

Proof: Given $\varepsilon > 0$, one has

$$ \begin{aligned} & \int_0^t \frac{E_{\alpha,\alpha}(-\varepsilon(t-s)^\alpha)}{(t-s)^{1-\alpha}} s^{-\alpha} ds \\ &= \sum_{k=0}^{\infty} \frac{(-\varepsilon)^k}{\Gamma(\alpha k+\alpha)} \int_0^t (t-s)^{\alpha k+\alpha-1} s^{-\alpha} ds \\ &= \sum_{k=0}^{\infty} \frac{(-\varepsilon)^k t^{\alpha k+\alpha-1-\alpha+1}}{\Gamma(\alpha k+\alpha)} \int_0^1 (1-\tau)^{\alpha k+\alpha-1} \tau^{-\alpha} d\tau \\ &= \sum_{k=0}^{\infty} \frac{(-\varepsilon)^k}{\Gamma(\alpha k+\alpha)} B(\alpha k+\alpha, 1-\alpha) t^{\alpha k} \\ &= \Gamma(1-\alpha) \sum_{k=0}^{\infty} \frac{(-\varepsilon)^k t^{\alpha k}}{\Gamma(\alpha k+1)} \\ &= \Gamma(1-\alpha)E_\alpha(-\varepsilon t^\alpha), \end{aligned} \tag{8}$$

where $B(p,q) = \int_0^1 (1-t)^{p-1}t^{q-1}dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ represents the Beta function. This completes the proof. ■

Lemma 3: [24] Let $\phi : [0,\infty) \to \mathbf{R}$ be a differentiable function. Then, for any given $t \geqslant 0$,

$$ \frac{1}{2} {}_0^C D_t^{\alpha} \phi^2(t) \leqslant \phi(t) {}_0^C D_t^{\alpha} \phi(t),\ \alpha \in (0,1). \tag{9}$$

III. Event-driven controller design and implementation

In this section, from a practical point of view, the control input u is assumed to be not continuously implemented, but is updated at certain instants $\{t_k\}_{k\geqslant 1}$, which form a sequence of strictly increasing positive instants to be specified later.

In the sequel, for simplicity, we write $A := \triangle + \sigma(x)$ with $\mathrm{Dom}(A) = \{\phi \in L^2(0,1) : \phi, \phi'$ are absolutely continuous and $\phi(0) = \phi(1) = 0\}$.

Consider system (1), if the control inputs are updated in event-driven cases, it can be can be reformulated as follows

$$\begin{cases} {}_0^C D_t^\alpha y(x,t) = Ay(x,t) \text{ in } (0,1)\times[t_k,t_{k+1}), \\ y(0,t) = 0 \text{ in } [t_k,t_{k+1}), \\ y(1,t) = u(t_k) + d(t) \text{ in } [t_k,t_{k+1}), \\ y(x,0) = y_0(x) \text{ in } (0,1). \end{cases} \tag{10}$$

A. Equivalent transform via backstepping

Similar to the argument in [25], the integral transformation

$$\omega(x,t) = y(x,t) - \int_0^x g(x,\xi)y(\xi,t)d\xi \tag{11}$$

with $\omega_0(x) = y_0(x) - \int_0^x g(x,\xi)y_0(\xi)d\xi$ is adopted to transform the system (10) into a target system. Then we obtain the following proposition, whose proof can be found in Appendix $V-A$.

Proposition 1: Suppose that the kernel $g(x,\xi)$ is chosen satisfying

$$\begin{cases} g_{xx}(x,\xi) - g_{\xi\xi}(x,\xi) = (\sigma(\xi)-\lambda)g(x,\xi),\ 0<\xi<x<1, \\ 2\frac{d}{dx}g(x,x) = \lambda - \sigma(x),\ 0<x<1, \\ g(x,0) = 0,\ 0<\xi<1, \end{cases} \tag{12}$$

the integral transformation (11) can equivalently convert the error dynamic (10) into

$$\begin{cases} {}_0^C D_t^\alpha \omega(x,t) = \triangle\omega(x,t) - \lambda\omega(x,t) \\ \qquad\qquad \text{in } (0,1)\times[t_k,t_{k+1}), \\ \omega(0,t) = 0 \text{ in } [t_k,t_{k+1}), \\ \omega(1,t) = u(t_k) + d(t) - \int_0^1 g(1,\xi)y(\xi,t)d\xi \\ \qquad \text{in } [t_k,t_{k+1}), \\ \omega(x,0) = \omega_0(x) \text{ in } (0,1). \end{cases} \tag{13}$$

B. Event-driven Controller design

Before designing the event-driven controller, we first give an estimation of the disturbance following the idea of uncertainty and disturbance estimator [13], [14], which has been extended to study the parabolic DPSs [15].

Consider the boundary conditions of systems (10) and (13), let

$$\begin{aligned} d(t) &= y(1,t) - u(t_k) \\ &= \omega(1,t) + \textstyle\int_0^1 g(1,\xi)y(\xi,t)d\xi - u(t_k). \end{aligned} \tag{14}$$

To estimate the disturbance, we introduce a low-pass filter as follows:

$$\begin{aligned} \hat{d}(t) &= \mathscr{L}^{-1}\{F(s)\}(t) * d(t) \\ &= \mathscr{L}^{-1}\{F(s)\}(t) * (y(1,t) - u(t_k)), \end{aligned} \tag{15}$$

where $\hat{d}$ denotes the estimation of d, $\mathscr{L}^{-1}$ is the inverse Laplace operator, $*$ represents the convolution operator and F satisfying

$$\lim_{s\to 0} s(1-F(s)) = 0 \tag{16}$$

is a low-pass filter in the frequency domain.

Design the event-driven boundary controller as

$$u(t_k) = \textstyle\int_0^1 g(1,\xi)y(\xi,t_k)d\xi - \hat{d}(t_k), \tag{17}$$

it follows that

$$\begin{aligned} u(t_k) &= \textstyle\int_0^1 g(1,\xi)y(\xi,t_k)d\xi - \hat{d}(t_k) + \hat{d}(t) - \hat{d}(t) \\ &= \textstyle\int_0^1 g(1,\xi)y(\xi,t_k)d\xi - \hat{d}(t_k) + \hat{d}(t) \\ &\quad - \mathscr{L}^{-1}\{F(s)\}(t) * (y(1,t) - u(t_k)) \\ &= \mathscr{L}^{-1}\left\{\tfrac{1}{1-F(s)}\right\}(t) * \textstyle\int_0^1 g(1,\xi)y(\xi,t_k)d\xi \\ &\quad - \mathscr{L}^{-1}\left\{\tfrac{1}{1-F(s)}\right\}(t) * \left(\hat{d}(t_k) - \hat{d}(t)\right) \\ &\quad - \mathscr{L}^{-1}\left\{\tfrac{F(s)}{1-F(s)}\right\}(t) * y(1,t). \end{aligned} \tag{18}$$

Let $\tilde{d}(t) = d(t) - \hat{d}(t)$. Substituting the controller (17) into the boundary condition of system (13), it yields that

$$\begin{aligned} \omega(1,t) &= \tilde{d}(t) + \textstyle\int_0^1 g(1,\xi)(y(\xi,t_k) - y(\xi,t))d\xi \\ &\quad - \hat{d}(t_k) + \hat{d}(t). \end{aligned} \tag{19}$$

Assume that the first event happens at $t_0 = 0$. Since $E_\alpha(-t^\alpha) \in (0,1)$ when $t \geqslant 0$. We design that the next instant t_k, $k = 1,2,\cdots$ is determined by

$$t_{k+1} = \min\left\{t > t_k : \frac{\delta_u(t_k,t)}{E_\alpha(-t^\alpha)} \geqslant \breve{e}\right\}, \tag{20}$$

where $\breve{e}$ is the event threshold,

$$\delta_u(t_k,t) = \|y(\cdot,t_k) - y(\cdot,t)\| + \left|\hat{d}(t_k) - \hat{d}(t)\right| \tag{21}$$

and $\{t_k\}_{k\in\mathbb{N}}$ represents the event-driven instants to show when the actuator signal is updated.

The following theorem gives the stability of considered closed-loop system.

Theorem 1: Suppose that (16) and all conditions of Lemma 3, Proposition 1 are satisfied. If both $\omega_x(1,t)$ and $d(t)$ are bounded, the Laplace transforms of $\omega^2(\cdot,t)$, $\omega_x(\cdot,t)$ and d exit and are finite. Then the closed loop system (1) under the event-driven rule (20) is asymptotically stable.

Proof: Note that system (10) can be equivalently converted to (13) via the inevitable integral transform (11) if the boundary controller is chosen as (17). From the statements in Appendix $V-A$, if $\sigma \in C^1[0,1]$, there exists a positive constant ν such that

$$\|y(\cdot,t)\| \leqslant \nu\|\omega(\cdot,t)\|,\ \|\omega_0\| \leqslant \nu\|y_0\|. \tag{22}$$

In what follows, we therefore, focus on studying the stability of system (13) with

$$\begin{aligned} \omega(1,t) &= \tilde{d}(t) + \textstyle\int_0^1 g(1,\xi)(y(\xi,t_k) - y(\xi,t))d\xi \\ &\quad - \hat{d}(t_k) + \hat{d}(t) \end{aligned} \tag{23}$$

under the event-driven rule (20).

Since $d(t)$ is bounded and $\mathscr{L}\{d\}(s)$ is finite, by (15),

$$\begin{aligned}&\lim_{s\to 0} s\mathscr{L}\{\tilde{d}\}(s)=\lim_{s\to 0} s\mathscr{L}\{d-\hat{d}\}(s)\\&=\lim_{s\to 0} s(1-F(s))\mathscr{L}\{d\}(s)=0.\end{aligned} \tag{24}$$

It then follows from the final value theorem that $\lim_{t\to\infty}\tilde{d}(t)=0$.

Consider that $\omega(\cdot,t)$ is a differentiable function, let

$$W(t)=\tfrac{1}{2}\int_0^1\omega(x,t)^2dx. \tag{25}$$

Lemma 3 yields that

$$\begin{aligned}{}_0^CD_t^\alpha W(t)&=\tfrac{1}{2}\int_0^1{}_0^CD_t^\alpha\omega(x,t)^2dx\\&\leqslant\int_0^1\omega(x,t){}_0^CD_t^\alpha\omega(x,t)dx\\&=\int_0^1\omega(x,t)\omega_{xx}(x,t)dx-\lambda\int_0^1\omega(x,t)^2dx\\&=\omega_x(1,t)\tilde{d}(t)+\omega_x(1,t)\left[\hat{d}(t)-\hat{d}(t_k)\right]\\&\quad+\omega_x(1,t)\int_0^1 g(1,\xi)(y(\xi,t_k)-y(\xi,t))d\xi\\&\quad-\int_0^1\omega_x(x,t)^2dx-\lambda\int_0^1\omega(x,t)^2dx\\&\leqslant\omega_x(1,t)\tilde{d}(t)+\omega_x(1,t)\rho(t)-2\lambda W(t),\end{aligned} \tag{26}$$

where

$$\rho(t)=\left[\hat{d}(t)-\hat{d}(t_k)\right]+\int_0^1 g(1,\xi)(y(\xi,t_k)-y(\xi,t))d\xi.$$

Let

$$M(t)=\omega_x(1,t)(\tilde{d}(t)+\rho(t))-2\lambda W(t)-{}_0^CD_t^\alpha W(t)\geqslant 0. \tag{27}$$

Since $\mathscr{L}(\omega^2)(\cdot,s)$ exists and $\omega(x,t)$ is the solution of system (13), taking the Laplace transform on both sides of (27) gives

$$\begin{aligned}\mathscr{L}(M)(s)&=\mathscr{L}(\omega_x(1,\cdot)(\tilde{d}+\rho))(s)-2\lambda\mathscr{L}(W)(s)\\&\quad-s^\alpha\mathscr{L}(W)(s)+s^{\alpha-1}W(0),\end{aligned} \tag{28}$$

where $W(0)=\frac{1}{2}\int_0^1\omega(x,0)^2dx\geqslant 0$. Hence,

$$\mathscr{L}(W)(s)=\frac{s^{\alpha-1}W(0)+\mathscr{L}(\omega_x(1,\cdot)(\tilde{d}+\rho))(s)-\mathscr{L}(M)(s)}{s^\alpha+2\lambda}.$$

It then follows from the uniqueness, existence theorem [23] that the unique solution of (27) is

$$\begin{aligned}W(t)&=E_\alpha(-2\lambda t^\alpha)W(0)\\&\quad+\left[\omega_x(1,t)(\tilde{d}(t)+\rho(t))\right]*\left[t^{\alpha-1}E_{\alpha,\alpha}(-2\lambda t^\alpha)\right]\\&\quad-M(t)*\left[t^{\alpha-1}E_{\alpha,\alpha}(-2\lambda t^\alpha)\right].\end{aligned} \tag{29}$$

Moreover, since $t^{\alpha-1}$ and $E_{\alpha,\alpha}(-2\lambda t^\alpha)$ are two nonnegative functions,

$$\begin{aligned}W(t)&\leqslant E_\alpha(-2\lambda t^\alpha)W(0)\\&\quad+\left[\omega_x(1,t)(\tilde{d}(t)+\rho(t))\right]*\left[\frac{E_{\alpha,\alpha}(-2\lambda t^\alpha)}{t^{1-\alpha}}\right].\end{aligned} \tag{30}$$

The boundedness of $\omega_x(1,t)$ implies that there exits a constant such that $|\omega_x(1,t)|\leqslant M_\omega$. From the event-driven rule (20), Lemma 1 and 2, we obtain that

$$\begin{aligned}&[\omega_x(1,t)\rho(t)]*\left[t^{\alpha-1}E_{\alpha,\alpha}(-2\lambda t^\alpha)\right]\\&=\int_0^t\frac{E_{\alpha,\alpha}(-2\lambda(t-s)^\alpha)}{(t-s)^{1-\alpha}}\omega_x(1,s)\rho(s)ds\\&\leqslant\breve{e}\max\{C_g,1\}M_\omega\int_0^t\frac{E_{\alpha,\alpha}(-2\lambda(t-s)^\alpha)}{(t-s)^{1-\alpha}}E_\alpha(-s^\alpha)ds\\&\leqslant\breve{e}\max\{C_g,1\}MM_\omega\int_0^t\frac{E_{\alpha,\alpha}(-2\lambda(t-s)^\alpha)}{(t-s)^{1-\alpha}}s^{-\alpha}ds\\&\leqslant\breve{e}\max\{C_g,1\}MM_\omega\Gamma(1-\alpha)E_\alpha(-2\lambda t^\alpha),\end{aligned} \tag{31}$$

where $C_g=\max|g(x,\xi)|$ is a constant defined as in Appendix $V-A$. Moreover, $\lim_{t\to\infty}\tilde{d}(t)=0$ and

$$\begin{aligned}&\lim_{t\to\infty}\int_0^t\frac{E_{\alpha,\alpha}(-2\lambda\tau^\alpha)}{\tau^{1-\alpha}}d\tau=\lim_{t\to\infty}\sum_{k=0}^{\infty}\int_0^t\frac{(-2\lambda)^k\tau^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)}d\tau\\&=\lim_{t\to\infty}\frac{1-E_\alpha(-2\lambda t^\alpha)}{2\lambda}=\frac{1}{2\lambda}\end{aligned} \tag{32}$$

imply that $t^{\alpha-1}E_{\alpha,\alpha}(-2\lambda\tau^\alpha)\in L^1[0,\infty)$ and

$$\tilde{d}(t)*\left[\frac{E_{\alpha,\alpha}(-2\lambda t^\alpha)}{t^{1-\alpha}}\right]\to 0 \text{ as } t\to\infty. \tag{33}$$

This is true following from the fact that the convolution of an L^1 function with a function tending to zero does, itself, tend to zero [26]. Observing that $E_\alpha(-2\lambda t^\alpha)\to 0$ as $t\to\infty$. We conclude that system (1) is asymptotically stable under the event-driven rule (20) and the proof is finished. ■

C. Minimum inter-event time

To avoid the Zeno phenomenon, a positive lower bounded minimum inter-event time should be guaranteed [27].

Theorem 2: Suppose that all conditions of Theorem 1 hold. Then the minimum inter-event time $T_{\min}$ given by

$$T_{\min}=\min_{k=0,1,2,\cdots}\{t_{k+1}-t_k\} \tag{34}$$

is lower bounded provided that t_k is defined as (20).

Proof: For any $k=0,1,2,\cdots$, let

$$\hat{e}(x,t)=y(x,t)-y(x,t_k), \tag{35}$$

where $t\in[t_k,t_{k+1})$. It follows that $\hat{e}(x,t_k)=\hat{e}(x,0)=0$. Consider system (10), the definition of Caputo fractional derivative lead to

$${}_0^CD_t^\alpha\hat{e}(x,t)={}_0^CD_t^\alpha y(x,t)=Ay(x,t)=A\hat{e}(x,t)+Ay(x,t_k)$$

with boundary conditions $\hat{e}_x(0,t)=0$, $\hat{e}(1,t)=d(t)-d(t_k)$ and initial condition $\hat{e}_0(x)=0$.

Based on the solution expression in [7], [28], where the solution of system (36) can be given using the spectral theory of operator A, i.e., one has

$$\begin{aligned}\hat{e}(x,t)&=\sum_{n=1}^{\infty}\int_0^t\frac{E_{\alpha,\alpha}((\sigma(x)+\lambda_n)\tau^\alpha)}{\tau^{1-\alpha}}d\tau(Ay(\cdot,t_k),\xi_n)\xi_n(x)\\&\quad+\sum_{n=1}^{\infty}\int_0^t\frac{E_{\alpha,\alpha}((\sigma(x)+\lambda_n)(t-\tau)^\alpha)}{(t-\tau)^{1-\alpha}}(d(\tau)-d(t_k))d\tau\frac{\partial\xi_n}{\partial x}(1)\xi_n(x).\end{aligned}$$

Moreover, since σ is positive and large enough, without loss of generality, suppose that $\sigma^*=\max_{x\in[0,1]}\sigma(x)>\pi^2$. By [29], since $E_{\alpha,\alpha}(t^\alpha)$ is an increasing function, it follows from the event-driven rule (20) that $|d(\tau)-d(t_k)|\leqslant\breve{e}E_\alpha(-\tau^\alpha)$, $\tau\in[t_k,t_{k+1})$ and then

$$\begin{aligned}\|\hat{e}(\cdot,t)\|&\leqslant\int_0^t\frac{E_{\alpha,\alpha}((b^*-\pi^2)\tau^\alpha)}{\tau^{1-\alpha}}d\tau\|Ay(\cdot,t_k)\|\\&\quad+\sqrt{2}\breve{e}\left\|\sum_{n=1}^{\infty}\int_0^t\frac{E_{\alpha,\alpha}((b(x)-\lambda_n)\tau^\alpha)}{\tau^{\alpha-1}}d\tau\left(n-\tfrac{1}{2}\right)\pi\xi_n(x)\right\|\\&\leqslant\frac{t^\alpha}{\alpha}E_{\alpha,\alpha}((b^*-\pi^2)t^\alpha)\|Ay(\cdot,t_k)\|\\&\quad+\sqrt{2}\breve{e}\pi\left[E_\alpha((b^*-\pi^2)t^\alpha)-1\right]\left(\sum_{n=1}^{\infty}\frac{\left(n-\frac{1}{2}\right)^2}{\left(n^2\pi^2-b^*\right)^2}\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore, by (20), a lower bound time $T_* > 0$ can be found such that

$$\|Ay(\cdot,t_k)\|\frac{E_{\alpha,\alpha}((b^*-\pi^2)T_*^{\alpha})}{\alpha E_\alpha(-T_*{}^{\alpha})}T_*^{\alpha} + \frac{\sqrt{2}\check{e}\pi\left(\sum_{n=1}^{\infty}\frac{\left(n-\frac{1}{2}\right)^2}{\left(n^2\pi^2-b^*\right)^2}\right)^{1/2}}{E_\alpha(-T_*{}^{\alpha})}\left[E_\alpha((b^*-\pi^2)T_*^{\alpha})-1\right] = \check{e}. \quad (36)$$

This implies that the minimum inter-event time $T_{\min}(T_{\min} \geqslant T_*)$ is lower bounded and the proof is finished. ■

IV. Numerical Example

In system (1), let $\alpha = 0.5$, $\sigma(x) = 15$ and $y_0(x) = \frac{x(1-x)}{2}e^{-x}$. By $15 \geqslant \pi^2$, then system (1) with $u \equiv 0$ is unstable. Let $\lambda = 1$ be in the target system (13). From the Chapter 4 of [30], the solution of hyperbolic PDE (12) is given by

$$m(x,\xi) = -14\xi\frac{I_1\left(\sqrt{14(x^2-\xi^2)}\right)}{\sqrt{14(x^2-\xi^2)}}, \quad (37)$$

where I_1 represents the modified Bessel functions of order one.

To test our event-driven control method, let $d(t) = \sin(t)$ and $\check{e} = 0.005$. We plot the spatial L^2-norm of the solution $y(x,t)$ to system (1) with event-trigged controller in 1) of Fig. 1. Moreover, 2) of Fig. 1 shows the event-driven instants when the control input is updated. It then can be seen that the event-driven scheme can significantly asymptotically stabilize the considered system.

V. Conclusions

In this paper, the idea of using state feedback event-driven control to asymptotically stabilize anomalous subdiffusion processes governed by time fractional diffusion systems with unknown time-varying input disturbance is presented. To address the problems, a UDE-based estimator is proposed to evaluate the unknown time-varying input disturbance through filtering. The main stability results are then obtained using a state feedback event-driven control strategic via Backstepping technique.

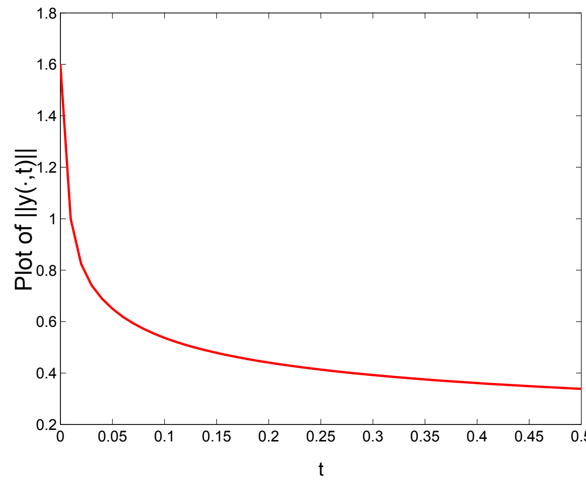
However, in many practical cases, the availability of full-state measurements may be impossible due to the difficulties in measuring. To solve this limitation, some new methods should be introduced to estimate the whole states of studied system. While interesting, we shall discuss these problems in our forthcoming work. Moreover, the results studied here can also be extended to more complex nonlinear fractional DPSs and various open questions are still under consideration. For more potential challenging topics concerning fractional DPSs, we refer the readers to [31] and the references therein.

Appendix.

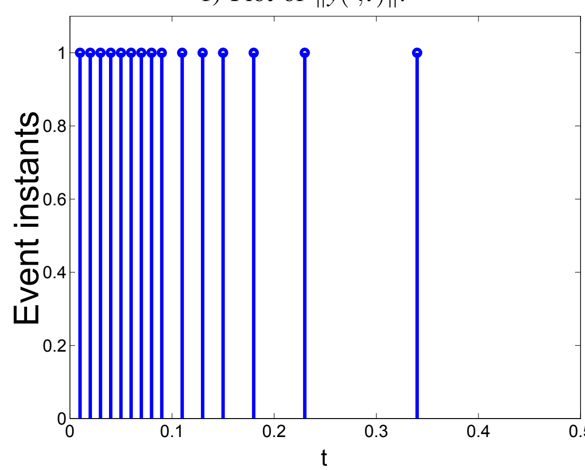
A. Proof of Proposition 1

Proof: Denote $g_\xi(x,x) = \frac{\partial}{\partial\xi}g(x,\xi)|_{\xi=x}$, $g_x(x,x) = \frac{\partial}{\partial x}g(x,\xi)|_{\xi=x}$ and $\frac{d}{dx}g(x,x) = g_x(x,x) + g_\xi(x,x)$. Differentiating (11) with respect to x, we see

$$\omega_x(x,t) = y_x(x,t) + g(x,x)y(x,t) + \int_0^x g_x(x,\xi)y(\xi,t)d\xi \quad (38)$$



1) Plot of $\|y(\cdot,t)\|$.



2) Event-driven instants.

Fig. 1. Evolution of solution to system (1) with event-trigged controller and the Event-driven instants.

and

$$\omega_{xx}(x,t) = y_{xx}(x,t) - \tfrac{d}{dx}g(x,x)y(x,t) - g(x,x)y_x(x,t) - g_x(x,x)y(x,t) - \textstyle\int_0^x g_{xx}(x,\xi)y(\xi,t)d\xi. \quad (39)$$

Similar to the argument in [25], by

$$\textstyle\int_0^x g(x,\xi)y_{\xi\xi}(\xi,t)d\xi = \int_0^x g_{\xi\xi}(x,\xi)y(\xi,t)d\xi + g(x,x)y_x(x,t) - g(x,0)y_x(0,t) - g_\xi(x,x)y(x,t) + g_\xi(x,0)y(0,t), \quad (40)$$

${}_0^C D_t^{\alpha} y(x,t) = Ay(x,t)$ and $y(0,t) = 0$, it follows that

$$\begin{aligned}0 &= {}_0^C D_t^{\alpha}\omega(x,t) - \omega_{xx}(x,t) - \lambda\omega(x,t)\\ &= {}_0^C D_t^{\alpha}y(x,t) - \textstyle\int_0^x g(x,\xi){}_0^C D_t^{\alpha}y(\xi,t)d\xi - y_{xx}(x,t)\\ &\quad + [\textstyle\int_0^x g(x,\xi)y(\xi,t)d\xi]_{xx} - \lambda y(x,t)\\ &\quad + \lambda\textstyle\int_0^x g(x,\xi)y(\xi,t)d\xi\\ &= {}_0^C D_t^{\alpha}y(x,t) - y_{xx}(x,t) - \textstyle\int_0^x g_{\xi\xi}(x,\xi)y(\xi,t)d\xi\\ &\quad - g(x,x)y_x(x,t) + g(x,0)y_x(0,t) + g_\xi(x,x)y(x,t)\\ &\quad - g_\xi(x,0)y(0,t) - \textstyle\int_0^x g(x,\xi)\sigma(\xi)y(\xi,t)d\xi\\ &\quad + \tfrac{d}{dx}g(x,x)y(x,t) + g(x,x)y_x(x,t) + g_x(x,x)y(x,t)\\ &\quad + \textstyle\int_0^x g_{xx}(x,\xi)y(\xi,t)d\xi - \lambda y(x,t)\\ &\quad + \lambda\textstyle\int_0^x g(x,\xi)y(\xi,t)d\xi\end{aligned}$$

$$= \int_0^x y(\xi,t)\left\{g_{xx}(x,\xi) - g_{\xi\xi}(x,\xi) + (\lambda - \sigma(\xi))g(x,\xi)\right\}d\xi$$
$$+ y(x,t)\left(\sigma(x) - \lambda + g_{\xi}(x,x) + \frac{d}{dx}g(x,x) + g_x(x,x)\right)$$
$$+ y_x(0,t)g(x,0).$$

The boundary conditions yield that $\omega(0,t) = y(0,t) = 0$ and

$$\omega(1,t) = y(1,t) - \int_0^1 g(1,\xi)y(\xi,t)d\xi. \quad (41)$$

This, together with $g_x(x,x) + g_{\xi}(x,x) = \frac{d}{dx}g(x,x)$, implies that if $g(x,\xi)$ is chosen satisfying (12), we get system (13).

Next, we show both the integral transformation (11) and its inverse are bounded.

By [30], the following existence results holds.

Lemma 4: [30] If $\sigma \in C^1[0,1]$, then system (12) has a unique bounded solution which is twice continuously differentiable in $0 < \xi < x < 1$.

Define $G : L^2(0,1) \to L^2(0,1)$ as

$$\omega(x,\cdot) = (Gy)(x,\cdot) := y(x,\cdot) - \int_0^x g(x,\xi)y(\xi,\cdot)d\xi, \quad (42)$$

where $g(x,\xi)$ is the solution of (12). By Lemma 4, since g is bounded, denote $C_g = \max|g(x,\xi)|$, it is not difficult to get that G is a bounded operator.

Set $\varphi(x,t) = \int_0^x g(x,\xi)y(\xi,t)d\xi$. Then $\omega(x,t) = y(x,t) - \varphi(x,t)$ and

$$\varphi(x,t) = \int_0^x g(x,\xi)\omega(\xi,t)d\xi + \int_0^x g(x,\xi)\varphi(\xi,t)d\xi. \quad (43)$$

Let $\varphi_0(x,t) = \int_0^x g(x,\xi)\omega(\xi,t)d\xi$ and $\varphi_n(x,t) = \int_0^x g(x,\xi)\varphi_{n-1}(\xi,t)d\xi$. We obtain that

$$|\varphi_0(x,t)| \leqslant C_g\|\omega(\cdot,t)\|, \ |\varphi_1(x,t)| \leqslant C_g^2\|\omega(\cdot,t)\|x, \quad (44)$$

$$|\varphi_2(x,t)| \leqslant \frac{C_g^3\|\omega(\cdot,t)\|}{2!}x^2, \ |\varphi_n(x,t)| \leqslant \frac{C_g^{n+1}\|\omega(\cdot,t)\|}{n!}x^n. \quad (45)$$

Therefore, the series $\varphi(x,t) = \sum_{n=0}^{\infty}\varphi_n(x,t)$ is absolutely and uniformly convergent and that its solution of (43). This means that G^{-1} exists and is bounded. The proof is completed. ■

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