Nonlinear Dyn (2018) 94:1523–1534
https://doi.org/10.1007/s11071-018-4439-z

ORIGINAL PAPER

New integral inequalities and asymptotic stability of fractional-order systems with unbounded time delay

Bin-Bin He · Hua-Cheng Zhou · Chun-Hai Kou · YangQuan Chen

Received: 31 March 2018 / Accepted: 18 June 2018 / Published online: 30 June 2018

Abstract The stability analysis of fractional-order systems with unbounded delay remains an open problem. In this paper, we firstly explore two new integral inequalities. Using these two integral inequalities obtained, the Halanay inequality with unbounded delay is extended to Caputo fractional-order case and Riemann–Liouville fractional-order case. Finally, several examples are presented to illustrate the effectiveness and applicability of the fractional Halanay inequalities in obtaining the asymptotic stability conditions of fractional-order systems with unbounded delay.

Keywords Asymptotic stability · Integral inequality · Fractional Halanay inequality · Unbounded time delay

B.-B. He
College of Information Science and Technology, Donghua University, Shanghai 201620, People's Republic of China
e-mail: hebinbin45@126.com

H.-C. Zhou (✉)
School of Mathematics and Statistics, Central South University, Changsha 410075, People's Republic of China
e-mail: hczhou@amss.ac.cn

C.-H. Kou
Department of Applied Mathematics, Donghua University, Shanghai 201620, People's Republic of China
e-mail: kouchunhai@dhu.edu.cn

Y. Chen
Mechatronics, Embedded Systems and Automation Lab, University of California, Merced, CA 95343, USA
e-mail: ychen53@ucmerced.edu

1 Introduction

Fractional calculus as an old mathematical topic is developed long ago by the mathematicians Leibniz, Liouville, Riemann and so on. It does not attract great attention due to the difficulty in computing and the uncertainty of its geometric significance. However, since these decades, researchers found that fractional calculus can precisely describe some abnormal phenomena, it is widely used in many areas such as physics, control and engineering, see [1–6].

As an important aspect of control systems, stability has attracted increasing interests. In 1996, Matignon [7] firstly presents the stability result of fractional-order linear system which can be used to determine the stability through the location in the complex plane of the dynamic matrix eigenvalues of the state. Since then, many results about the stability of fractional-order systems are obtained. The main methods to analyze the stability of fractional-order linear system include Laplace transform method [8], linear matrix inequality (LMI) approach [9] and the Riesz basis approach and the semigroup method [10]. For the stability of fractional-order nonlinear system, the main approaches include the comparison method [11,12], the integral inequality method [13], the linearization technique [14] and the Lyapunov method [15,16].

The Lyapunov direct method is a powerful tool to analyze the stability of integer-order nonlinear systems which can be verified easily without solving the system.

For fractional-order nonlinear system, the Lyapunov method is not developed until 2009 [15] and 2010 [16], where the authors proposed the Mittag-Leffler stability, which is a generalization of classical exponential stability, and they explored the fractional Lyapunov method. Nevertheless, since the well-known Leibniz chain rule is invalid for fractional-order derivative, the application of Lyapunov method is not available until 2014 [17] where the authors proposed a simple but useful derivative inequality, which makes x^2 become a good Lyapunov candidate function. A similar differential inequality in the Riemann–Liouville sense was explored in 2016 [18].

Time delay as a common phenomenon is often encountered in different real systems, such as electric, synchronous, chemical processes [19–21]. It is worth to remark that, the existence of delay may cause some undesirable response, even instability, so the research on the stability of fractional-order system with time delay is meaningful and important. In [22,23], the authors generalized the Razumikhin method to fractional-order systems with bounded delay. Using frequency domain method, the asymptotic stability results about fractional retarded and neutral systems are obtained in [24,25]. In [18,26], the authors explored the fractional-order systems with unbounded delay and they obtained the stability of the systems with the restriction $\dot{\tau}(t) \leq d < 1$. The stability results in [18,26,27] are essentially in L^2 norm sense. It is noted that the convergence in L^2 norm sense does not imply the convergence in pointwise sense, i.e., $x(t) \to 0$ as $t \to \infty$ cannot follow from $\int_{t-\tau(t)}^{t} x^2(s)\mathrm{d}s \to 0$ as $t \to \infty$.

In this paper, motivated mainly by [18,26,27], we focus on the asymptotic stability of fractional-order systems with unbounded delay, where the fractional-order derivatives are in Caputo sense and Riemann–Liouville sense. The main tools proposed here which are also our first contribution are two integral inequalities which generalize the Halanay inequality with bounded delay to the fractional Halanay inequality with unbounded delay. The second novelty is that we do not need the assumption that $\dot{\tau}(t) \leq d < 1$ used in [18,26] (also see Remark 3).

The rest of this paper is organized as follows: Sect. 2 presents some basic concepts and lemmas on fractional calculus. The new integral inequalities and fractional Halanay inequalities with unbounded delay are introduced in Sect. 3. Several kinds of fractional-order systems with unbounded delay are explored in Sect. 4 where the asymptotic stability conditions are obtained. Section 5 is a conclusion about this paper.

2 Preliminary

In this section, some basic definitions and lemmas are presented which are useful throughout this paper.

Definition 1 [2] The Riemann–Liouville fractional integral of order $\alpha > 0$ for a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$ {}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s. $$

Based on this definition of Riemann–Liouville fractional integral, the fractional-order derivative in Riemann–Liouville sense and Caputo sense are given.

Definition 2 [2] The Riemann–Liouville fractional derivative of order $\alpha > 0$ for a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$ \begin{aligned} {}_0^{RL}D_t^{\alpha} f(t) &= \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left({}_0D_t^{-(k-\alpha)} f(t)\right) \\ &= \frac{1}{\Gamma(k-\alpha)} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \int_0^t (t-s)^{k-\alpha-1} f(s)\mathrm{d}s, \\ &\quad t > 0, \end{aligned} $$

where $k-1 \leq \alpha < k$, $k \in \mathbb{N}$ and $\Gamma(\cdot)$ is the Gamma function, that is

$$ \Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t}\mathrm{d}t. $$

In particular, when $0 < \alpha < 1$, we have

$$ {}_0^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-\alpha} f(s)\mathrm{d}s. $$

Definition 3 [2] The Caputo fractional derivative of order $\alpha > 0$ for a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$ \begin{aligned} {}_0^{C}D_t^{\alpha} f(t) &= {}_0D_t^{-(k-\alpha)} f^{(k)}(t) \\ &= \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-s)^{k-\alpha-1} f^{(k)}(s)\mathrm{d}s, \\ &\quad t > 0, \end{aligned} $$

where $k-1 \leq \alpha < k$, $k \in \mathbb{N}$ and $f^{(m)}(t)$ is the m-order derivative of $f(t)$. In particular, when $0 < \alpha < 1$, we have

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} \mathrm{d}s. $$

A very important concept in the theory of fractional calculus is Mittag-Leffler function.

Definition 4 [2] The two-parameter Mittag-Leffler function is defined by

$$ E_{\alpha,\beta}(z) = \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(\alpha j+\beta)}, \quad \alpha > 0, \quad \beta > 0. $$

When $\beta = 1$, the two-parameter Mittag-Leffler function becomes the one-parameter Mittag-Leffler function, i.e.,

$$ E_\alpha(z) = E_{\alpha,1}(z) = \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(\alpha j+1)}, \ \alpha > 0. $$

It is well known that the computation of the fractional derivatives for composite functions is complicated since there is no chain rule for fractional-order derivative. That is, generally, ${}_0^C D_t^\alpha f(x(t)) \neq f'(x(t)) {}_0^C D_t^\alpha x(t)$. This leads to the difficulty of computation ${}_0^C D_t^\alpha x^2(t)$. Fortunately, for the sake of stability, the following two lemmas about fractional-order derivative inequalities are enough to analyze the stability of a class of fractional systems considered in this paper.

Lemma 1 [17] *Let $\alpha \in (0,1)$ and $x(t) \in \mathbb{R}$ be a continuous and differentiable function. Then for any $t \geq 0$,* ${}_0^C D_t^\alpha x^2(t) \leq 2x(t) {}_0^C D_t^\alpha x(t)$.

Lemma 2 [18] *Let $\alpha \in (0,1)$ and $x(t) \in \mathbb{R}$ be a continuous and differentiable function. Then for any $t \geq 0$,* ${}_0^{RL} D_t^\alpha x^2(t) \leq x(t) {}_0^{RL} D_t^\alpha x(t)$.

Consider the following fractional-order system with unbounded delay

$$ {}_0 D_t^\alpha x(t) = f(t, x(t), x(t-\tau(t))), \quad t \geq 0, \tag{1} $$

where $0 < \alpha < 1$, ${}_0 D_t^\alpha x(t)$ represents the fractional-order derivative of $x(t)$ in Riemann–Liouville or Caputo sense. $x(t) \in \mathbb{R}^n$ is the state vector, $f: \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear functions with $f(0,0,0) = 0$. $\tau(t): [0,+\infty) \to [0,+\infty)$ represents the unbounded time delay.

Definition 5 For fractional-order system (1), the trivial solution is called to be asymptotically stable if $x(t) \to 0$ as $t \to +\infty$.

3 Integral inequalities

In this section, we establish two integral inequalities which can be used to deal with the asymptotic stability for fractional-order systems with unbounded time-varying delay. The proof is based on "inf–sup" method.

Theorem 1 *Let $\phi: [-h,+\infty) \to \mathbb{R}^+$ be bounded on $[-h,0]$ and continuous on $[0,+\infty)$. Suppose that $a \in C(\mathbb{R}^+,\mathbb{R})$ satisfies $\lim_{t\to+\infty} a(t) = 0$, $\tau \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\tau(t) \leq t+h$ and $t-\tau(t) \to +\infty$ as $t \to +\infty$. $K \in C(\mathbb{R}^+,\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, and $\lim_{t\to+\infty} K(t) = 0$. $\mu > 0$ is a constant and the following inequality holds: for all $t \geq 0$,*

$$ \phi(t) \leq a(t) + \mu \int_0^t K(t-s) \sup_{s-\tau(s)\leq\sigma\leq s} \phi(\sigma) \mathrm{d}s. \tag{2} $$

If $\mu \|K\|_{L^1(\mathbb{R}^+)} < 1$, then, $\lim_{t\to+\infty} \phi(t) = 0$.

Proof The proof is divided into two steps.

Step 1 Proving that $\phi(t)$ is uniformly bounded on $[-h,+\infty)$.

Indeed, since $\tau(t) \leq t+h$, for $s \in [0,t]$, we have

$$ \sup_{s-\tau(s)\leq\sigma\leq s} \phi(\sigma) \leq \sup_{\sigma\in[-h,t]} \phi(\sigma) $$

which, combining with (2), gives

$$ \begin{aligned} \phi(t) &\leq a(t) + \sup_{\sigma\in[-h,t]} \phi(\sigma) \mu \int_0^t K(t-s) \mathrm{d}s \\ &= a(t) + \sup_{\sigma\in[-h,t]} \phi(\sigma) \mu \int_0^t K(s) \mathrm{d}s \\ &\leq a(t) + \sup_{\sigma\in[-h,t]} \phi(\sigma) \mu \|K\|_{L^1(\mathbb{R}^+)}, \end{aligned} \tag{3} $$

Taking the supremum on both sides of (3) yields

$$ \begin{aligned} \sup_{\sigma\in[-h,t]} \phi(\sigma) \leq{}& \sup_{s\geq 0} a(s) + \sup_{\sigma\in[-h,0]} \phi(\sigma) \\ &+ \sup_{\sigma\in[-h,t]} \phi(\sigma) \mu \|K\|_{L^1(\mathbb{R}^+)}. \end{aligned} \tag{4} $$

Since $\mu \|K\|_{L^1(\mathbb{R}^+)} < 1$, it follows from (4) that for all $t \geq 0$,

$$ \sup_{\sigma\in[-h,t]} \phi(\sigma) \leq \frac{\sup_{s\geq 0} a(s) + \sup_{\sigma\in[-h,0]} \phi(\sigma)}{1 - \mu \|K\|_{L^1(\mathbb{R}^+)}}, \tag{5} $$

that is, $\phi(t)$ is bounded on $[-h,+\infty)$.

Step 2 Showing that $\lim_{t\to+\infty}\phi(t)=0$. Denote $\overline{\phi}(t):=\sup_{s\geq t}\phi(s)$. It is clear to see that $\overline{\phi}(t)$ is well defined and is non-increasing with respect to t since $\phi(t)$ is nonnegative and is bounded on $[-h,+\infty)$ just proved in step 1, which also implies the existence of $\inf_{t\geq 0}\overline{\phi}(t)$. Thus, to order to show $\lim_{t\to+\infty}\phi(t)=0$, it suffices to prove $\inf_{t\geq 0}\overline{\phi}(t)=0$.

Indeed, for any given $\varepsilon>0$, there exists a constant $T\geq 0$ such that $\overline{\phi}(t)\leq\overline{\phi}(T)\leq\inf_{t\geq 0}\overline{\phi}(t)+\varepsilon$ holds for all $t\geq T$. Since $\lim_{t\to+\infty}(t-\tau(t))=+\infty$, there exist $\overline{T}>0$ such that for all $t\geq\overline{T}$, $t-\tau(t)>T$ holds. It follows from (2) and (5) that

$$\begin{aligned}
\phi(t) &\leq a(t)+\mu\int_0^t K(t-s)\overline{\phi}(s-\tau(s))\mathrm{d}s\\
&\leq a(t)+\mu\int_0^{\overline{T}} K(t-s)\overline{\phi}(s-\tau(s))\mathrm{d}s\\
&\quad+\mu\int_{\overline{T}}^t K(t-s)\overline{\phi}(s-\tau(s))\mathrm{d}s\\
&\leq a(t)+\frac{\sup_{s\geq 0}a(s)+\sup_{\sigma\in[-h,0]}\phi(\sigma)}{1-\mu\|K\|_{L^1(\mathbb{R}^+)}}\mu\\
&\quad\times\int_0^{\overline{T}}K(t-s)\mathrm{d}s+\mu\left[\inf_{t\geq 0}\overline{\phi}(t)+\varepsilon\right]\int_{\overline{T}}^t K(t-s)\mathrm{d}s\\
&\leq a(t)+\frac{\sup_{s\geq 0}a(s)+\sup_{\sigma\in[-h,0]}\phi(\sigma)}{1-\mu\|K\|_{L^1(\mathbb{R}^+)}}\mu\\
&\quad\times\int_0^{\overline{T}}K(t-s)\mathrm{d}s+\left[\inf_{t\geq 0}\overline{\phi}(t)+\varepsilon\right]\mu\|K\|_{L^1(\mathbb{R}^+)}.
\end{aligned}\tag{6}$$

By the definition of $\overline{\phi}(t)$ and (6), we get that for $t\geq\overline{T}$,

$$\begin{aligned}
\overline{\phi}(t) &\leq \sup_{s\geq t}\left[a(s)+\frac{\sup_{s\geq 0}a(s)+\sup_{\sigma\in[-h,0]}\phi(\sigma)}{1-\mu\|K\|_{L^1(\mathbb{R}^+)}}\right.\\
&\quad\left.\times\mu\int_0^{\overline{T}}K(s-\sigma)\mathrm{d}\sigma+\left[\inf_{t\geq 0}\overline{\phi}(t)+\varepsilon\right]\mu\|K\|_{L^1(\mathbb{R}^+)}\right]\\
&\leq \sup_{s\geq t}a(s)+\frac{\sup_{s\geq 0}a(s)+\sup_{\sigma\in[-h,0]}\phi(\sigma)}{1-\mu\|K\|_{L^1(\mathbb{R}^+)}}\\
&\quad\times\mu\int_0^{\overline{T}}\sup_{s\geq t}K(s-\sigma)\mathrm{d}\sigma+\left[\inf_{t\geq 0}\overline{\phi}(t)+\varepsilon\right]\mu\|K\|_{L^1(\mathbb{R}^+)}.
\end{aligned}\tag{7}$$

From $\lim_{t\to+\infty}a(t)=0$ and $\lim_{t\to+\infty}K(t)=0$, one has

$$\lim_{t\to+\infty}\left\{\sup_{s\geq t}a(s)+\frac{\sup_{s\geq 0}a(s)+\sup_{\sigma\in[-h,0]}\phi(\sigma)}{1-\mu\|K\|_{L^1(\mathbb{R}^+)}}\times\mu\int_0^{\overline{T}}\sup_{s\geq t}K(s-\sigma)\mathrm{d}\sigma\right\}=0.$$

Taking the infimum on both sides of (7) and noting that $\inf_{t\geq 0}\overline{\phi}(t)=\lim_{t\to+\infty}\overline{\phi}(t)$, we derive

$$\inf_{t\geq 0}\overline{\phi}(t)\leq\left[\inf_{t\geq 0}\overline{\phi}(t)+\varepsilon\right]\mu\|K\|_{L^1(\mathbb{R}^+)},$$

which, jointly with $\mu\|K\|_{L^1(\mathbb{R}^+)}<1$, yields $\inf_{t\geq 0}\overline{\phi}(t)\leq\varepsilon\mu\|K\|_{L^1(\mathbb{R}^+)}/(1-\mu\|K\|_{L^1(\mathbb{R}^+)})$. The arbitrariness of ε implies $\inf_{t\geq 0}\overline{\phi}(t)=0$. Proof is completed. □

Theorem 2 *Let $\phi\colon(0,+\infty)\to\mathbb{R}^+$ be continuous. Suppose that $a\colon(0,+\infty)\to\mathbb{R}$ is a continuous function satisfying $\lim_{t\to+\infty}a(t)=0$. $\tau\colon(0,+\infty)\to(0,+\infty)$ is continuous and satisfies $t-\tau(t)\geq g(t)>0$ for all $t>0$, where $g(t)$ is monotonically increasing and $\lim_{t\to+\infty}g(t)=+\infty$. $K\in C(\mathbb{R}^+,\mathbb{R}^+)\cap L^1(\mathbb{R}^+)$, satisfies $\lim_{t\to+\infty}K(t)=0$, and there exists a constant $T>0$ such that $\int_0^T K(t-s)\sup_{s-\tau(s)\leq\sigma\leq s}\phi(\sigma)\mathrm{d}s$ is bounded for $t>g(T)$ and tends to 0 as $t\to+\infty$. $\mu>0$ is a constant and the following inequality holds: for all $t\geq 0$,*

$$\phi(t)\leq a(t)+\mu\int_0^t K(t-s)\sup_{s-\tau(s)\leq\sigma\leq s}\phi(\sigma)\mathrm{d}s.\tag{8}$$

If $\mu\|K\|_{L^1(\mathbb{R}^+)}<1$, then, $\lim_{t\to+\infty}\phi(t)=0$.

Proof We prove this theorem by two steps.

Step 1 Showing that $\phi(t)$ is uniformly bounded for all $t>g(T)$.

Noting that $0<g(t)\leq t-\tau(t)$, it follows from (8) that

$$\begin{aligned}
\phi(t) &\leq a(t)+\mu\int_0^T K(t-s)\sup_{s-\tau(s)\leq\sigma\leq s}\phi(\sigma)\mathrm{d}s\\
&\quad+\mu\int_T^t K(t-s)\sup_{g(s)\leq s-\tau(s)\leq\sigma\leq s}\phi(\sigma)\mathrm{d}s\\
&\leq a(t)+\mu\int_0^T K(t-s)\sup_{s-\tau(s)\leq\sigma\leq s}\phi(\sigma)\mathrm{d}s\\
&\quad+\sup_{\sigma\in[g(T),t]}\phi(\sigma)\mu\int_T^t K(t-s)\mathrm{d}s\\
&\leq a(t)+\mu\int_0^T K(t-s)\sup_{s-\tau(s)\leq\sigma\leq s}\phi(\sigma)\mathrm{d}s\\
&\quad+\sup_{\sigma\in[g(T),t]}\phi(\sigma)\mu\|K\|_{L^1(\mathbb{R}^+)},
\end{aligned}\tag{9}$$

which implies

$$\sup_{t\geq g(T)}\phi(t)\leq\sup_{t\geq g(T)}a(t)+\mu\sup_{t\geq g(T)}\int_0^T K(t-s)\times\sup_{s-\tau(s)\leq\sigma\leq s}\phi(\sigma)\mathrm{d}s+\sup_{t\geq g(T)}\phi(t)\mu\|K\|_{L^1(\mathbb{R}^+)}. \tag{10}$$

Since $a(t)$ is continuous and $\lim_{t\to+\infty}a(t)=0$, it is seen that $a(t)$ is bounded on $[g(T),+\infty)$. Since

$$\int_0^T K(t-s)\sup_{s-\tau(s)\leq\sigma\leq s}\phi(\sigma)\mathrm{d}s$$

is bounded and $\mu\|K\|_{L^1(\mathbb{R}^+)}<1$, it follows from (10) that

$$\sup_{t\geq g(T)}\phi(t)\leq\frac{\sup_{t\geq g(T)}a(t)+\mu\sup_{t\geq g(T)}\int_0^T K(t-s)\sup_{s-\tau(s)\leq\sigma\leq s}\phi(\sigma)\mathrm{d}s}{1-\mu\|K\|_{L^1(\mathbb{R}^+)}}:=N,\ t\geq g(T), \tag{11}$$

that is, $\phi(t)$ is uniformly bounded on $[g(T),+\infty)$.

Step 2 Showing that $\lim_{t\to+\infty}\phi(t)=0$. Denote $\overline{\phi}(t):=\sup_{s\geq t>g(T)}\phi(s)$, it is seen that $\overline{\phi}(t)$ is well defined and is non-increasing with respect to t since $\phi(t)$ is nonnegative and is bounded on $[g(T),+\infty)$ just proved in step 1, which also implies the existence of $\inf_{t>g(T)}\overline{\phi}(t)$. Thus, in order to show $\lim_{t\to+\infty}\phi(t)=0$, it suffices to prove $\inf_{t>g(T)}\overline{\phi}(t)=0$. Indeed, by the definition of infimum, for any given $\varepsilon>0$, there exists $G>g(T)$ such that $\overline{\phi}(t)\leq\overline{\phi}(G)\leq\inf_{t>g(T)}\overline{\phi}(t)+\varepsilon$ for all $t\geq G$. Considering the fact that $0<g(t)\leq t-\tau(t)$ and $g(t)$ is increasing, we have $g(T)<T$ and $g^{-1}(g(G))=G<g^{-1}(G)$. It follows from (8) and (11) that

$$\begin{aligned}\phi(t)&\leq a(t)+\mu\int_0^T K(t-s)\sup_{s-\tau(s)\leq\sigma\leq s}\phi(\sigma)\mathrm{d}s\\&\quad+\mu\int_T^{g^{-1}(G)}K(t-s)\sup_{s-\tau(s)\leq\sigma\leq s}\phi(\sigma)\mathrm{d}s\\&\quad+\mu\int_{g^{-1}(G)}^t K(t-s)\sup_{g(s)\leq s-\tau(s)\leq\sigma\leq s}\phi(\sigma)\mathrm{d}s\\&\leq a(t)+\mu\int_0^T K(t-s)\sup_{s-\tau(s)\leq\sigma\leq s}\phi(\sigma)\mathrm{d}s\\&\quad+\mu N\int_T^{g^{-1}(G)}K(t-s)\mathrm{d}s\\&\quad+\mu\sup_{[G,t]}\phi(\sigma)\int_{g^{-1}(G)}^t K(t-s)\mathrm{d}s\\&\leq a(t)+\mu\int_0^T K(t-s)\sup_{s-\tau(s)\leq\sigma\leq s}\phi(\sigma)\mathrm{d}s\\&\quad+\mu N\int_T^{g^{-1}(G)}K(t-s)\mathrm{d}s\\&\quad+\left[\inf_{t>g(T)}\overline{\phi}(t)+\varepsilon\right]\mu\|K\|_{L^1(\mathbb{R}^+)}\end{aligned} \tag{12}$$

By the definition of $\overline{\phi}(t)$ and (12), we get that for $t\geq g^{-1}(G)>g(T)$,

$$\begin{aligned}\overline{\phi}(t)&\leq\sup_{s\geq t}\Bigg[a(s)+\mu\int_0^T K(s-\sigma)\sup_{\sigma-\tau(\sigma)\leq\omega\leq\sigma}\phi(\omega)\mathrm{d}\sigma\\&\quad+\mu N\int_T^{g^{-1}(G)}K(s-\sigma)\mathrm{d}\sigma\\&\quad+\left[\inf_{t>g(T)}\overline{\phi}(t)+\varepsilon\right]\mu\|K\|_{L^1(\mathbb{R}^+)}\Bigg]\\&\leq\sup_{s\geq t}a(s)+\mu\sup_{s\geq t}\int_0^T K(s-\sigma)\sup_{\sigma-\tau(\sigma)\leq\omega\leq\sigma}\phi(\omega)\mathrm{d}\sigma\\&\quad+\mu N\int_T^{g^{-1}(G)}\sup_{s\geq t}K(s-\sigma)\mathrm{d}\sigma\\&\quad+\left[\inf_{t\geq T}\overline{\phi}(t)+\varepsilon\right]\mu\|K\|_{L^1(\mathbb{R}^+)}.\end{aligned} \tag{13}$$

From $\lim_{t\to+\infty}a(t)=0$ and $\lim_{t\to+\infty}K(t)=0$, one has

$$\lim_{t\to+\infty}\Bigg\{\sup_{s\geq t}a(s)+\mu\sup_{s\geq t}\int_0^T K(s-\sigma)\times\sup_{\sigma-\tau(\sigma)\leq\omega\leq\sigma}\phi(\omega)\mathrm{d}\sigma+\mu N\int_T^{g^{-1}(G)}\sup_{s\geq t}K(s-\sigma)\mathrm{d}\sigma\Bigg\}=0.$$

Taking the infimum on both sides of (13) and noting that $\inf_{t>g(T)}\overline{\phi}(t)=\lim_{t\to+\infty}\overline{\phi}(t)$, we derive

$$\inf_{t>g(T)}\overline{\phi}(t)\leq\left[\inf_{t>g(T)}\overline{\phi}(t)+\varepsilon\right]\mu\|K\|_{L^1(\mathbb{R}^+)},$$

which, jointly with $\mu\|K\|_{L^1(\mathbb{R}^+)}<1$, yields $\inf_{t>g(T)}\overline{\phi}(t)\leq\varepsilon\mu\|K\|_{L^1(\mathbb{R}^+)}/(1-\mu\|K\|_{L^1(\mathbb{R}^+)})$. The arbitrariness of ε yields $\inf_{t>g(T)}\overline{\phi}(t)=0$. Proof is completed. □

By using Theorem 1, we arrive at the following corollary.

Corollary 1 *Let* $\alpha \in (0,1)$ *and* $V\colon [-h, +\infty) \to \mathbb{R}^+$ *be bounded on* $[-h, 0]$ *and continuous on* $[0, +\infty)$. *Suppose that* $\tau \in C(\mathbb{R}^+, \mathbb{R}^+)$ *satisfies* $\tau(t) \le t + h$ *for some fixed* $h > 0$, $t - \tau(t) \to +\infty$ *as* $t \to +\infty$. *For some positive constants* $\lambda > \mu > 0$, *the following inequality holds: for all* $t \ge 0$,

$$ {}_0^C D_t^\alpha V(t) \le -\lambda V(t) + \mu \sup_{-\tau(t)\le\sigma\le 0} V(t+\sigma). \tag{14}$$

Then $\lim_{t\to+\infty} V(t) = 0$.

Proof According to (14), for $t \ge 0$, there exists a non-negative function $M(t)$ satisfying

$$ {}_0^C D_t^\alpha V(t) + M(t) = -\lambda V(t) + \mu \sup_{-\tau(t)\le\sigma\le 0} V(t+\sigma). \tag{15}$$

Taking the Laplace transform on both sides of (15) yields, for $t \ge 0$,

$$ s^\alpha \widehat{V}(s) - V(0)s^{\alpha-1} + \widehat{M}(s) = -\lambda \widehat{V}(s) + \mu F(s), \tag{16}$$

where

$$\begin{aligned} \widehat{V}(s) &:= \int_0^\infty e^{-st} V(t)\mathrm{d}t, \\ \widehat{M}(s) &:= \int_0^\infty e^{-st} M(t)\mathrm{d}t, \\ F(s) &:= \int_0^\infty e^{-st} \sup_{-\tau(t)\le\sigma\le 0} V(t+\sigma)\mathrm{d}t, \end{aligned}$$

are, respectively, the Laplace transform of the functions $V(t)$, $M(t)$ and $\sup_{-\tau(t)\le\sigma\le 0} V(t+\sigma)$. Therefore, by (16), it has $\widehat{V}(s)$ the expression given by

$$ \widehat{V}(s) = \frac{V(0)s^{\alpha-1} - \widehat{M}(s) + \mu F(s)}{s^\alpha + \lambda}. \tag{17}$$

Taking the inverse Laplace transform on both sides of (17) gives

$$\begin{aligned} V(t) &= E_\alpha(-\lambda t^\alpha)V(0) - M(t) * \left[t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)\right] \\ &\quad + \mu \left[\sup_{-\tau(t)\le\sigma\le 0} V(t+\sigma)\right] * \left[t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)\right], \end{aligned} \tag{18}$$

where $*$ represents the convolution operator. Since $M(t)$, $t^{\alpha-1}$ and $E_{\alpha,\alpha}(-\lambda t^\alpha)$ are nonnegative, we have the following estimate:

$$\begin{aligned} V(t) &\le E_\alpha(-\lambda t^\alpha)V(0) + \mu \int_0^t (t-s)^{\alpha-1} \\ &\quad \times E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) \sup_{-\tau(t)\le\sigma\le 0} V(s+\sigma)\mathrm{d}s. \end{aligned} \tag{19}$$

It can be easily seen that (19) is the form of (2) with $a(t) = E_\alpha(-\lambda t^\alpha)V(0)$ and $K(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$. Obviously, $a(\cdot)$ is continuous and $\lim_{t\to+\infty} a(t) = 0$. From Theorem 1, in order to prove $\lim_{t\to+\infty} V(t) = 0$, it is sufficient to verify $\mu\|K\|_{L^1(\mathbb{R}^+)} < 1$. Indeed, it follows from [3, p. 50, formula 1.10.7] that

$$\frac{\mathrm{d}}{\mathrm{d}t}[t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha)] = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha),$$

which implies

$$\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda s^\alpha)\mathrm{d}s = t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha). \tag{20}$$

It follows from [3, Page 43, formula 1.8.28] that

$$E_{\alpha,\alpha+1}(-\lambda t^\alpha) = \frac{1}{\lambda t^\alpha} + \mathcal{O}\left(\frac{1}{\lambda^2 t^{2\alpha}}\right),$$

which yields

$$\lim_{t\to+\infty} t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) = \frac{1}{\lambda}.$$

Since $\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda s^\alpha)\mathrm{d}s$ is non-decreasing with respect to t, from (20), so does for $t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha)$. Thus,

$$\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda s^\alpha)\mathrm{d}s \le \frac{1}{\lambda}, \quad \forall t \ge 0. \tag{21}$$

which, jointly with $\lambda > \mu$, implies

$$\mu\|K\|_{L^1(\mathbb{R}^+)} = \mu \int_0^\infty s^{\alpha-1}E_{\alpha,\alpha}(-\lambda s^\alpha) \le \frac{\mu}{\lambda} < 1. \tag{22}$$

Hence $\lim_{t\to+\infty} V(t) = 0$. □

By using Theorem 2, we arrive at the following corollary.

Corollary 2 *Let* $\alpha \in (0,1)$ *and* $V\colon (0, +\infty) \to \mathbb{R}^+$ *be continuous on* $(0, +\infty)$ *and* $t^{1-\alpha}V(t)$ *is continuous* *on* $[0, +\infty)$. $\tau\colon (0, +\infty) \to (0, +\infty)$ *is continuous and satisfies* $t - \tau(t) \geq g(t) > 0$ *for all* $t > 0$, *where* $g(t)$ *is monotonically increasing,* $\lim_{t\to+\infty} g(t) = +\infty$ *and* $(g(t))^{\alpha-1}$ *is integrable on* $(0, T]$. *Assume that for some positive constants* $\lambda > \mu > 0$, *the following inequality holds: for all* $t \geq 0$,

$$
{}_0^{RL}D_t^{\alpha} V(t) \leq -\lambda V(t) + \mu \sup_{-\tau(t)\leq\sigma\leq 0} V(t+\sigma). \tag{23}
$$

Then $\lim_{t\to+\infty} V(t) = 0$.

Proof By (23), for $t > 0$, we know that there exists a nonnegative function $M(t)$ satisfying

$$
{}_0^{RL}D_t^{\alpha} V(t) + M(t) = -\lambda V(t) + \mu \sup_{-\tau(t)\leq\sigma\leq 0} V(t+\sigma). \tag{24}
$$

Taking the Laplace transform on both sides of (24) yields

$$
\begin{aligned}
& s^{\alpha}\widehat{V}(s) - {}_0^{RL}D_t^{-(1-\alpha)} V(0) + \widehat{M}(s) \\
& \quad = -\lambda \widehat{V}(s) + \mu F(s),
\end{aligned} \tag{25}
$$

where $t > 0$ and

$$
\begin{aligned}
\widehat{V}(s) &:= \int_0^{\infty} e^{-st} V(t)\mathrm{d}t, \\
\widehat{M}(s) &:= \int_0^{\infty} e^{-st} M(t)\mathrm{d}t, \\
F(s) &:= \int_0^{\infty} e^{-st} \sup_{-\tau(t)\leq\sigma\leq 0} V(t+\sigma)\mathrm{d}t,
\end{aligned}
$$

are, respectively, the Laplace transform of the functions $V(t)$, $M(t)$ and $\sup_{-\tau(t)\leq\sigma\leq 0} V(t+\sigma)$. Therefore, (25) is reduced to

$$
\widehat{V}(s) = \frac{{}_0^{RL}D_t^{-(1-\alpha)} V(0) - \widehat{M}(s) + \mu F(s)}{s^{\alpha} + \lambda}. \tag{26}
$$

Taking the inverse Laplace transform on both sides of (26) gives

$$
\begin{aligned}
V(t) &= t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha}) {}_0^{RL}D_t^{-(1-\alpha)} V(0) \\
&\quad - M(t) * [t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})] \\
&\quad + \mu \left[\sup_{-\tau(t)\leq\sigma\leq 0} V(t+\sigma) \right] * [t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})],
\end{aligned} \tag{27}
$$

where $*$ represents the convolution operator. Since $M(t)$, $t^{\alpha-1}$ and $E_{\alpha,\alpha}(-\lambda t^{\alpha})$ are nonnegative, it follows that

$$
\begin{aligned}
V(t) &\leq t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha}) {}_0^{RL}D_t^{-(1-\alpha)} V(0) \\
&\quad + \mu \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha}) \\
&\quad \times \sup_{-\tau(s)\leq\sigma\leq 0} V(s+\sigma)\mathrm{d}s.
\end{aligned} \tag{28}
$$

Clearly, (28) is the form of (8) with $K(t) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha})$ and $a(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha}) {}_0^{RL}D_t^{-(1-\alpha)} V(0)$, it can be seen that $a(\cdot)$ is continuous on $(0, +\infty)$ and satisfies $\lim_{t\to+\infty} a(t) = 0$. Similar to the proof of Corollary 1, we have $\lim_{t\to+\infty} K(t) = 0$ and $\mu\|K\|_{L^1(\mathbb{R}^+)} < 1$. To show $\lim_{t\to+\infty} V(t) = 0$, it suffices to prove that $\int_0^T K(t-s) \sup_{s-\tau(s)\leq\sigma\leq s} V(\sigma)\mathrm{d}s$ is bounded for $t > g(T)$ and tends to zero as $t \to +\infty$. Actually, for $t > g(T)$, it has

$$
\begin{aligned}
& \int_0^T K(t-s) \sup_{s-\tau(s)\leq\sigma\leq s} V(\sigma)\mathrm{d}s \\
& \quad = \int_0^T K(t-s) \sup_{g(s)\leq s-\tau(s)\leq\sigma\leq s} \sigma^{\alpha-1}\sigma^{1-\alpha} V(\sigma)\mathrm{d}s \\
& \quad \leq \int_0^T K(t-s)(g(s))^{\alpha-1} \sup_{s-\tau(s)\leq\sigma\leq s} \sigma^{1-\alpha} V(\sigma)\mathrm{d}s.
\end{aligned}
$$

Since $t^{1-\alpha}V(t)$ is continuous on $[0, T]$, we obtain

$$
\begin{aligned}
& \int_0^T K(t-s) \sup_{s-\tau(s)\leq\sigma\leq s} V(\sigma)\mathrm{d}s \\
& \quad \leq \int_0^T K(t-s)(g(s))^{\alpha-1}\mathrm{d}s \sup_{0\leq\sigma\leq T} \sigma^{1-\alpha}V(\sigma) \\
& \quad \leq \left(\int_0^{\frac{g(T)}{2}} K(t-s)(g(s))^{\alpha-1}\mathrm{d}s \right. \\
& \qquad \left. + \int_{\frac{g(T)}{2}}^T K(t-s)(g(s))^{\alpha-1}\mathrm{d}s \right) \sup_{0\leq\sigma\leq T} \sigma^{1-\alpha}V(\sigma) \\
& \quad \leq \left(\sup_{s\in\left[0, \frac{g(T)}{2}\right]} K(t-s) \int_0^{\frac{g(T)}{2}} (g(s))^{\alpha-1}\mathrm{d}s \right. \\
& \qquad \left. + \sup_{s\in\left[\frac{g(T)}{2}, T\right]} (g(s))^{\alpha-1} \int_{\frac{g(T)}{2}}^T K(t-s)\mathrm{d}s \right) \\
& \qquad \times \sup_{0\leq\sigma\leq T} \sigma^{1-\alpha}V(\sigma).
\end{aligned} \tag{29}
$$

Since $(g(s))^{\alpha-1}$ is integrable on $(0, T]$ and $\lim_{t\to\infty} K(t) = 0$, it can be verified that

$$\sup_{s\in\left[0,\frac{g(T)}{2}\right]} K(t-s)\int_0^{\frac{g(T)}{2}} (g(s))^{\alpha-1}\mathrm{d}s$$

is bounded for all $t > g(T)$ and converges to zero as $t \to \infty$. On the other hand, since $K(t)$ is integrable on $(0, T]$ and $\lim_{t\to\infty} K(t) = 0$, it is easy to check that

$$\sup_{s\in\left[\frac{g(T)}{2},T\right]} (g(s))^{\alpha-1}\int_{\frac{g(T)}{2}}^{T} K(t-s)\mathrm{d}s$$

is bounded for all $t > g(T)$ and converges to zero as $t \to \infty$. Thus, by (29), $\int_0^T K(t-s)(g(s))^{\alpha-1}\mathrm{d}s$ is bounded for all $t > g(T)$ and tends to zero as $t \to +\infty$. Hence, it follows from Theorem 2 that $\lim_{t\to+\infty} V(t) = 0$. □

Remark 1 Corollaries 1 and 2 can be regarded as generalization of Halanay inequality with unbounded delay [28]. Corollary 1 is applicable for the asymptotic stability of Caputo fractional-order system while Corollary 2 can be applied to obtain the stability of Riemann–Liouville fractional-order system. According to the characteristic of solution expression of fractional-order system ${}_0D_t^\alpha x(t) = -\lambda x(t) + f(x)$, where ${}_0D_t^\alpha$ represents Caputo derivative or Riemann–Liouville derivative, $V(t)$ in Corollaries 1 and 2 has different domain of $V(t)$ (One is $t \geq -h$, the other is $t > 0$), and thus $\tau(t)$ in Corollary 1 is supposed to satisfy $t - \tau(t) \geq -h$, whereas $\tau(t)$ in Corollary 2 is required to satisfy $t - \tau(t) > 0$ which is assumed in [18,26].

Remark 2 In [23], the authors use Razumikhin theorem to investigate the stability of fractional-order system with bounded delay. Corollaries 1 and 2 can be regarded as a generalization of Theorem 3.2 in [23].

4 Applications on asymptotic stability

In this section, by applying the inequalities derived in the previous section, we investigate the asymptotic stability for several classes of fractional-order systems with unbounded time-varying delay.

Example 1 Consider the following fractional-order differential system

$$ {}_0^CD_t^\alpha x(t) = -a(t)x(t) + b(t)x(t-\tau(t)), \tag{30}$$

where $0 < \alpha < 1$, $x \in \mathbb{R}$ is the state, $a(t), b(t)$ are continuous functions, $\tau(t)$ is a continuous function satisfying $t - \tau(t) \geq -h$ for $t \geq 0$ and $t - \tau(t) \to +\infty$ as $t \to +\infty$.

Denote $V(t) = x^2(t)$ and choose two constants $\lambda > \mu > 0$. Finding Caputo's derivative of $V(t)$ with respect to t along the solution to (30) yields

$$\begin{aligned}
&{}_0^CD_t^\alpha V(t) + \lambda V(t) - \mu \sup_{-\tau(t)\leq\sigma\leq 0} V(t+\sigma)\\
&\leq 2x(t){}_0^CD_t^\alpha x(t) + \lambda x^2(t) - \mu x^2(t-\tau(t))\\
&= 2x(t)[-a(t)x(t) + b(t)x(t-\tau)] + \lambda x^2(t)\\
&\quad - \mu x^2(t-\tau(t))\\
&= [-2a(t)+\lambda]x^2(t) + 2b(t)x(t)x(t-\tau(t))\\
&\quad - \mu x^2(t-\tau(t))\\
&= \begin{pmatrix} x(t) & x(t-\tau(t)) \end{pmatrix}\\
&\quad \begin{pmatrix} -2a(t)+\lambda & b(t) \\ b(t) & -\mu \end{pmatrix}\begin{pmatrix} x(t) \\ x(t-\tau(t)) \end{pmatrix}\\
&\leq 0,
\end{aligned}$$

provided that

$$\begin{pmatrix} -2a(t)+\lambda & b(t) \\ b(t) & -\mu \end{pmatrix} \leq 0, \quad \forall t \geq 0. \tag{31}$$

By Corollary 1, if (31) holds, the solution of (30) is asymptotically stable.

Proposition 1 *Suppose that there exist two constants $\lambda > \mu > 0$ such that LMI (31) holds. Then system (30) with $x_0 = \phi \in C([-h, 0], \mathbb{R})$ is asymptotically stable for all unbounded delay $\tau(t)$ satisfying $\tau(t) \leq t + h$ and $t - \tau(t) \to +\infty$ as $t \to +\infty$.*

When $a(t) = a$, $b(t) = b$ in (30) are two constants functions, (31) becomes

$$\begin{pmatrix} -2a+\lambda & b \\ b & -\mu \end{pmatrix} \leq 0. \tag{32}$$

This is equivalent to $\mu > 0$, $-2a+\lambda < 0$ and $-(-2a+\lambda)\mu - b^2 > 0$, that is, $\mu > 0$, $\lambda \in (0, 2a)$ and $(2a-\lambda)\mu > b^2$. It is easy to verify that if $a > |b|$, there exists two constants $\lambda, \mu > 0$ with $\lambda > \mu$ such that $\lambda \in (0, 2a)$ and $(2a-\lambda)\mu > b^2$, i.e., LMI (32) is feasible. Hence the following corollary is proved.

Corollary 3 *Suppose that $a > |b|$. Then system*

$$ {}_0^CD_t^\alpha x(t) = -ax(t) + bx(t-\tau(t)), \tag{33}$$

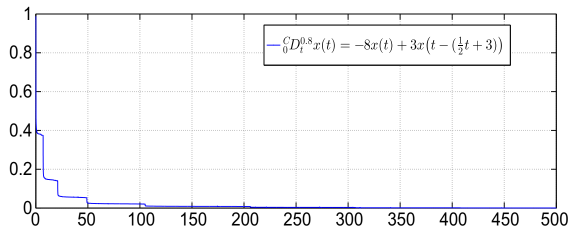

Fig. 1 System (33) with $\alpha = 0.8, a = 8, b = 3, x_0 = 1 \in C[-3, 0]$, and with delay $\tau(t) = t/2 + 3$

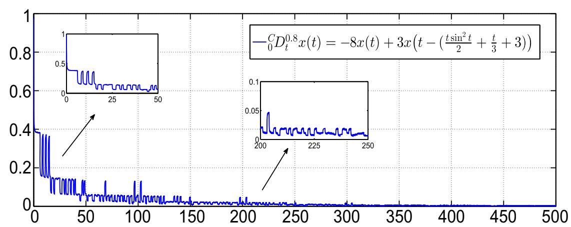

Fig. 2 System (33) with $\alpha = 0.8, a = 8, b = 3, x_0 = 1 \in C[-3, 0]$, and with delay $\tau(t) = t\sin^2(t)/2 + t/3 + 3$

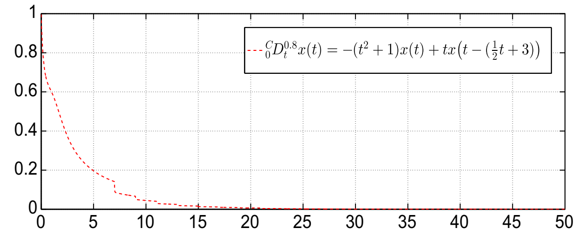

Fig. 3 System (30) with $\alpha = 0.8, a = t^2 + 1, b = t, x_0 = 1 \in C[-3, 0]$, and with delay $\tau(t) = t/2 + 3$

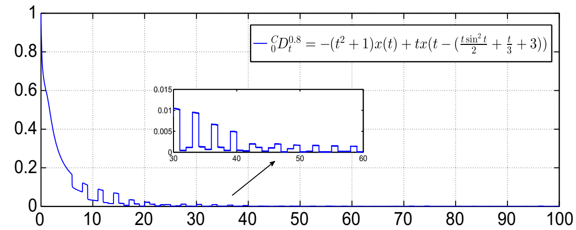

Fig. 4 System (30) with $\alpha = 0.8, a = t^2 + 1, b = t, x_0 = 1 \in C[-3, 0]$, and with delay $\tau(t) = t\sin^2 t/2 + t/3 + 3$

with $0 < \alpha < 1$, $x_0 = \phi \in C[-h, 0]$ *is asymptotically stable for all unbounded delay* $\tau(t)$ *satisfying* $\tau(t) \leq t + h$ *and* $t - \tau(t) \to +\infty$ *as* $t \to +\infty$.

Note that $t - \tau(t) \to +\infty$ as $t \to +\infty$ whence $\tau(t)$ is bounded. Corollary 3 generalizes the results of [13,29], where both time delays are required to be bounded.

Figure 1 gives a numerical simulation of system (33) with $\alpha = 0.8, a = 8, b = 3, \tau(t) = \frac{1}{2}t + 3$ and $x_0 = 1 \in C[-3, 0]$. Figure 2 gives a numerical simulation of system (33) with $\alpha = 0.8, a = 8, b = 3, \tau(t) = \frac{t\sin^2(t)}{2} + \frac{t}{3} + 3$ and $x_0 = 1 \in C[-3, 0]$.

Now taking $a(t) = t^2 + 1, b(t) = t$ and choosing $\lambda = 1, \mu = \frac{1}{2}$, a simple computation shows that LMI (31) holds for all $t \geq 0$. Hence system ${}_0^C D_t^{0.8}x(t) = -(t^2 + 1)x(t) + tx(t - \tau(t))$ is asymptotically stable with unbounded delay $\tau(t)$ satisfying $\tau(t) \leq t + h$ and $t - \tau(t) \to +\infty$ as $t \to +\infty$. The simulation of this system with delay $\tau(t) = \frac{1}{2}t + 3$ and $\tau(t) = \frac{t\sin^2(t)}{2} + \frac{t}{3} + 3$ can be seen in Figs. 3 and 4, respectively, where the initials are $x_0 = 1 \in C[-3, 0]$.

Example 2 Consider the following fractional-order nonlinear system with delay

$$\begin{aligned} {}_0^C D_t^{\alpha} x(t) &= f(x(t), y(t), z(t)) + ax(t - \tau(t)), \\ {}_0^C D_t^{\alpha} y(t) &= g(x(t), y(t), z(t)) + by(t - \tau(t)), \\ {}_0^C D_t^{\alpha} z(t) &= h(x(t), y(t), z(t)) + cz(t - \tau(t)). \end{aligned} \tag{34}$$

where $0 < \alpha < 1$, $x, y, z \in \mathbb{R}$ are states, $f, g, h: \mathbb{R}^3 \to \mathbb{R}$ are continuous functions, $\tau(t) > 0$ is a continuous function satisfying $t - \tau(t) \geq -h$ and $t - \tau(t) \to +\infty$ as $t \to +\infty$.

Proposition 2 *Suppose that there exist constants* λ, μ *such that* $\lambda > \mu > 0$ *and*

$$\Big(\frac{a^2}{\mu} + \lambda\Big)x^2 + \Big(\frac{b^2}{\mu} + \lambda\Big)y^2 + \Big(\frac{c^2}{\mu} + \lambda\Big)z^2 + 2xf(x, y, z) + 2yg(x, y, z) + 2zh(x, y, z) \leq 0$$

holds. Then system (34) *with* $x_0 = \phi \in C([-h, 0], \mathbb{R})$, $y_0 = \varphi \in C([-h, 0], \mathbb{R})$, $z_0 = \psi \in C([-h, 0], \mathbb{R})$ *is asymptotically stable for all unbounded delay* $\tau(t)$ *satisfying* $t - \tau(t) \geq -h$ *and* $t - \tau(t) \to +\infty$ *as* $t \to +\infty$.

Proof Denote $V(t) = x^2(t) + y^2(t) + z^2(t)$. Finding Caputo's derivative of $V(t)$ with respect to t along the solution to (34) gives

$$\begin{aligned} {}_0^C D_t^{\alpha} V(t) &\leq 2x(t){}_0^C D_t^{\alpha} x(t) + 2y(t){}_0^C D_t^{\alpha} y(t) \\ &\quad + 2z(t){}_0^C D_t^{\alpha} z(t) = 2x(t)\big(f(x(t), y(t), z(t)) \\ &\quad + ax(t - \tau(t))\big) \\ &\quad + 2y(t)\big(g(x(t), y(t), z(t)) + by(t - \tau(t))\big) \\ &\quad + 2z(t)\big(h(x(t), y(t), z(t)) + cz(t - \tau(t))\big) \end{aligned} \tag{35}$$

By Young's inequality and (2), it follows that

$$
\begin{aligned}
&{}_0^C D_t^\alpha V(t) + \lambda V(t) - \mu \sup_{-\tau(t)\le\sigma\le 0} V(t+\sigma)\\
&\le 2x(t)\big(f(x(t),y(t),z(t)) + ax(t-\tau(t))\big)\\
&\quad + 2y(t)\big(g(x(t),y(t),z(t)) + by(t-\tau(t))\big)\\
&\quad + 2z(t)\big(h(x(t),y(t),z(t)) + cz(t-\tau(t))\big) + \lambda x^2(t)\\
&\quad + \lambda y^2(t) + \lambda z^2(t) - \mu \sup_{-\tau(t)\le\sigma\le 0} x^2(t+\sigma)\\
&\quad - \mu \sup_{-\tau(t)\le\sigma\le 0} y^2(t+\sigma) - \mu \sup_{-\tau(t)\le\sigma\le 0} z^2(t+\sigma)\\
&\le 2x(t)f(x(t),y(t),z(t)) + \frac{a^2x^2(t)}{\mu} + \mu x^2(t-\tau(t))\\
&\quad + 2y(t)g(x(t),y(t),z(t)) + \frac{b^2y^2(t)}{\mu}\\
&\quad + \mu y^2(t-\tau(t)) + 2z(t)h(x(t),y(t),z(t)) + \frac{c^2z^2(t)}{\mu}\\
&\quad + \mu z^2(t-\tau(t)) + \lambda x^2(t) + \lambda y^2(t) + \lambda z^2(t)\\
&\quad - \mu x^2(t-\tau(t)) - \mu y^2(t-\tau(t)) - \mu z^2(t-\tau(t))\\
&= 2x(t)f(x(t),y(t),z(t)) + \frac{a^2x^2(t)}{\mu}\\
&\quad + 2y(t)g(x(t),y(t),z(t)) + \frac{b^2y^2(t)}{\mu}\\
&\quad + 2z(t)h(x(t),y(t),z(t)) + \frac{c^2z^2(t)}{\mu}\\
&\quad + \lambda x^2(t) + \lambda y^2(t) + \lambda z^2(t)\\
&\le 0
\end{aligned}
$$

which, jointly with Corollary 1, implies that $V(t) = x^2(t) + y^2(t) + z^2(t) \to 0$ as $t \to +\infty$. Thus, system (34) is asymptotically stable. □

Now taking $f(x,y,z) = -2x + y + y^2 + z$, $g(x,y,z) = -xy - x - 2y - z$, $h(x,y,z) = -x + y - 2z$, and choosing $\lambda = 2$, $\mu = 1$, $a = b = c = 1$, it is easy to verify that system (34) with $\alpha = 0.8$, initials $x_0 = y_0 = z_0 = 1 \in C[-3,0]$ is asymptotically stable for unbounded delay $\tau(t)$ satisfying $\tau(t) < t + h$ and $t - \tau(t) \to +\infty$ as $t \to +\infty$. The simulations of system (34) with the above parameters and $\tau(t) = \frac{1}{2}t + 3$ and $\tau(t) = \frac{t\sin^2(t)}{2} + \frac{t}{3} + 3$ can be seen in Figs. 5 and 6, respectively.

Example 3 Consider the following fractional-order nonlinear system in Riemann–Liouville sense with unbounded time delay

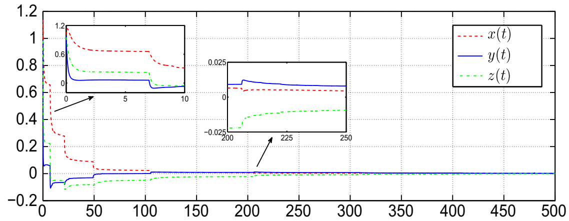

Fig. 5 System (34) with $\alpha = 0.8$, $x_0 = y_0 = z_0 = 1 \in C[-3,0]$, and with delay $\tau(t) = t/2 + 3$

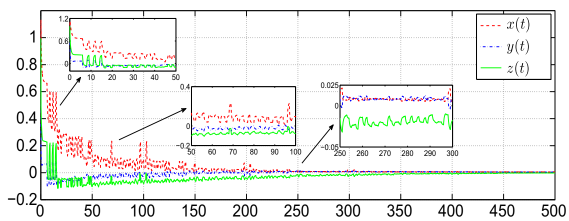

Fig. 6 System (34) with $\alpha = 0.8$, $x_0 = y_0 = z_0 = 1 \in C[-3,0]$, and with delay $\tau(t) = t\sin^2(t)/2 + t/3 + 3$

$$
\begin{aligned}
{}_0^{RL}D_t^\alpha x(t) &= -3x(t) - \frac{3}{2}x(t-\tau(t)) + \frac{1}{4}y(t-\tau(t))\\
&\quad + x(t)\sin(y(t)),\\
{}_0^{RL}D_t^\alpha y(t) &= -2y(t) + \frac{1}{2}y(t-\tau(t))\\
&\quad + y(t)\sin(x(t-\tau(t))),
\end{aligned}
\tag{36}
$$

where $0 < \alpha < 1$, $x(t), y(t) \in \mathbb{R}$. $\tau(t) > 0$ is a continuous function and satisfies $t - \tau(t) > 0$ and $t - \tau(t) \to +\infty$ as $t \to +\infty$

In order to prove the stability of system (36), we shall construct Lyapunov function $V(t) = x^2(t) + y^2(t)$ to verify that there exist $\lambda > \mu > 0$ such that inequality ${}_0^{RL}D_t^\alpha V(t) + \lambda V(t) - \mu \sup_{-\tau(t)\le\sigma\le 0} V(t+\sigma) \le 0$ holds. By Lemma 2, finding Riemann–Liouville derivative of $V(t)$ with respect to t along the solution to (36) gives:

$$
\begin{aligned}
&{}_0^{RL}D_t^\alpha V(t) + \lambda V(t) - \mu \sup_{-\tau(t)\le\sigma<0} V(t+\sigma)\\
&\le 2x(t)\Big[-3x(t) - \frac{3}{2}x(t-\tau(t)) + \frac{1}{4}y(t-\tau(t))\\
&\quad + x(t)\sin(y(t))\Big] + 2y(t)\Big[-2y(t) + \frac{1}{2}y(t-\tau(t))\\
&\quad + y(t)\sin(x(t-\tau(t)))\Big]\\
&\quad + \lambda x^2(t) + \lambda y^2(t) - \mu x^2(t-\tau(t)) - \mu y^2(t-\tau(t))\\
&= -6x^2(t) - 3x(t)x(t-\tau(t)) + \frac{1}{2}x(t)y(t-\tau(t))
\end{aligned}
$$

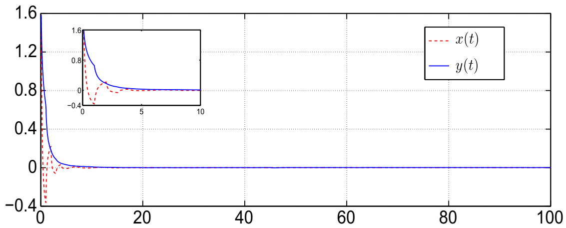

Fig. 7 System (34) with $\alpha = 0.8, {}_0D_t^{\alpha-1}x(t)|_{t\to 0} = {}_0D_t^{\alpha-1}y(t)|_{t\to 0} = 1$, and with delay $\tau(t) = t/5$

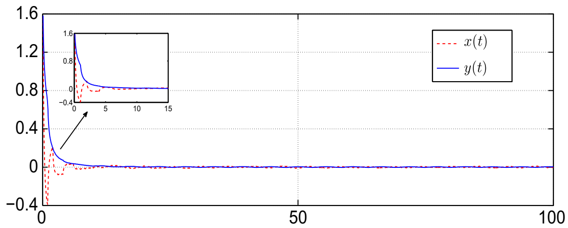

Fig. 8 System (34) with $\alpha = 0.8, {}_0D_t^{\alpha-1}x(t)|_{t\to 0} = {}_0D_t^{\alpha-1}y(t)|_{t\to 0} = 1$, and with delay $\tau(t) = t/4 + t\sin^2(10t)/4$

$$
\begin{aligned}
&+2x^2(t)\sin(y(t)) - 4y^2(t) + y(t)y(t-\tau(t))\\
&+2y^2(t)\sin(x(t-\tau(t))) + \lambda x^2(t) + \lambda y^2(t)\\
&-\mu x^2(t-\tau(t)) - \mu y^2(t-\tau(t))\\
\le\ &(\lambda-4)x^2(t) + (\lambda-2)y^2(t) - 3x(t)x(t-\tau(t))\\
&+\frac{1}{2}x(t)y(t-\tau(t)) + y(t)y(t-\tau(t))\\
&-\mu x^2(t-\tau(t)) - \mu y^2(t-\tau(t))\\
&=\begin{pmatrix} x(t) & y(t) & x(t-\tau(t)) & y(t-\tau(t)) \end{pmatrix}\\
&\times \begin{pmatrix} \lambda-4 & 0 & -3/2 & 1/4\\ 0 & \lambda-2 & 0 & 1/2\\ -3/2 & 0 & -\mu & 0\\ 1/4 & 1/2 & 0 & -\mu \end{pmatrix} \begin{pmatrix} x(t)\\ y(t)\\ x(t-\tau(t))\\ y(t-\tau(t)) \end{pmatrix}\\
\le\ &0
\end{aligned}
$$

if

$$
\begin{pmatrix} \lambda-4 & 0 & -3/2 & 1/4\\ 0 & \lambda-2 & 0 & 1/2\\ -3/2 & 0 & -\mu & 0\\ 1/4 & 1/2 & 0 & -\mu \end{pmatrix} \le 0 \tag{37}
$$

holds. It can be directly calculated that with $\lambda = 1.7$ and $\mu = 1.2$, LMI (37) holds and thus (36) is asymptotically stable. The evolution of system (36) with $\alpha = 0.8$ and initials ${}_0D_t^{\alpha-1}x(t)|_{t\to 0} = 1, {}_0D_t^{\alpha-1}y(t)|_{t\to 0} = 1$ can be seen in Figs. 7 and 8 where the delays are $\tau(t) = \frac{t}{5}$ and $\tau(t) = \frac{t}{4} + \frac{t\sin^2(10t)}{4}$, respectively.

Remark 3 In [18], the authors prove the stability result by using the Lyapunov method under the restriction $\dot{\tau}(t) < 1$. Also, the stability results of [18] is in L^2 norm sense. In this paper, we have removed the assumption that $\dot{\tau}(t) < 1$ by using the integral inequalities to obtain the asymptotic stability in $\mathbb{R}^n$ norm sense. Particularly, $\dot{\tau}(t) < 1$ is invalid for the case where $\tau(t) = \frac{t}{4} + \frac{t\sin^2(10t)}{4}$, the stability results derived here cannot be obtained from the method used in [18].

5 Conclusion

In this paper, we investigated the asymptotic stability of fractional-order systems with the Caputo fractional derivative and Riemann–Liouville fractional derivative. Firstly, two integral inequalities are presented, by which the Halanay inequality is generalized to fractional Halanay inequality with unbounded delay. According to the characteristics of the solution of fractional-order system, we made the assumptions that $t - \tau(t) \ge -h$ for Caputo fractional-order system and $t - \tau(t) > 0$ for Riemann–Liouville fractional-order system. Examples are included to illustrate the effectiveness of our results. It should be remarked that our results are easily applied to obtain the stability conditions.

In future works, applying the fractional Halanay inequality obtained in the paper to time fractional reaction diffusion equation with time-varying delay seems to be an interesting problem. At addition, a future research direction may be to use the fractional Halanay inequality and LMI approach in [30] to design the robust state feedback controller for fractional nonlinear system with bounded or unbounded time delay and to analyze the global stability analysis for fractional-order neural networks with mixed time-varying delays [12,31].

Acknowledgements This work is partially supported by the Fundamental Research Funds for the Central Universities (No. CUSF-DH-D-2017083).

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers, Amsterdam (1993)
2. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1998)
3. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
4. Sun, H.G., Chen, W., Chen, Y.Q.: Variable-order fractional differential operators in anomalous diffusion modeling. Phys. A **388**, 4586–4592 (2009)
5. Monje, C.A., Chen, Y.Q., Vinagre, B.M., Xue, D., Feliu-Batlle, V.: Fractional-Order Systems and Controls: Fundamentals and Applications. Springer, London (2010)
6. Mainardi, F.: Fractional calculus and Wave in Linear Viscoelasticity: An Introduction to Mathematical Models. World Scientific, Singapore (2010)
7. Matignon, D.: Stability results for fractional differential equations with applications to control processing. In: IMACS/IEEE-SMC Multiconference, Symposium on Control, Optimization and Supervision (CESA), pp. 963–968 (1996)
8. Deng, W.H., Li, C.P., Lü, J.H.: Stability analysis of linear fractional differential system with multiple time delays. Nonlinear Dyn. **48**, 409–416 (2007)
9. Sabatier, J., Moze, M., Farges, C.: LMI stability conditions for fractional order systems. Comput. Math. Appl. **59**, 1594–1609 (2010)
10. Zhou, H.C., Guo, B.Z.: Boundary feedback stabilization for an unstable time fractional reaction diffusion equation. SIAM J. Control Optim. **56**, 75–101 (2018)
11. Choi, S.K., Koo, N.: The monotonic property and stability of solutions of fractional differential equations. Nonlinear Anal. Theor. **74**, 6530–6536 (2011)
12. Wang, H., Yu, Y.G., Wen, G.G., Zhang, S., Yu, J.Z.: Global stability analysis of fractional-order Hopfield neural networks with time delay. Neurocomputing **154**, 15–23 (2015)
13. He, B.B., Zhou, H.C., Chen, Y.Q., Kou, C.H.: Asymptotical stability of fractional order systems with time delay via an integral inequality. IET Control Theory Appl. (2018). https://doi.org/10.1049/iet-cta.2017.1144
14. Lenka, B.K., Banerjee, S.: Sufficient conditions for asymptotic stability and stabilization of autonomous fractional order systems. Commun. Nonlinear Sci. Numer. Simul. **56**, 365–379 (2018)
15. Li, Y., Chen, Y.Q., Podlubny, I.: Mittag-Leffler stability of fractional order nonlinear dynamic systems. Automatica **45**, 1965–1969 (2009)
16. Li, Y., Chen, Y.Q., Podlubny, I.: Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability. Comput. Math. Appl. **59**, 1810–1821 (2010)
17. Aguila-Camacho, N., Duarte-Mermoud, M., Gallegos, J.A.: Lyapunov functions for fractional order systems. Commun. Nonlinear Sci. Numer. Simul. **19**, 2951–2957 (2014)
18. Liu, S., Wu, X., Zhou, X.F., Jiang, W.: Asymptotical stability of Riemann–Liouville fractional nonlinear systems. Nonlinear Dyn. **86**, 65–71 (2016)
19. Hale, J.K., Lunel, S.M.V.: Introduction to Functional Differential Equations. Springer, New York (1993)
20. Lakshmikantham, V., Wen, L., Zhang, B.: Theory of Differential Equations with Unbounded Delay. Kluwer, Dordrecht (1994)
21. Fridman, E.: Introduction to Time Delay Systems, Analysis and Control. Springer, Basel (2014)
22. Baleanu, D., Sadati, S.J., Ghaderi, R., Ranjbar, A., Abdeljawad, T., Jarad, F.: Razumikhin stability theorem for fractional systems with delay. Abstr. Appl. Anal. (2010). https://doi.org/10.1155/2010/124812
23. Chen, B., Chen, J.: Razumikhin-type stability theorems for functional fractional-order differential systems and applications. Appl. Math. Comput. **254**, 63–69 (2015)
24. Bonnet, C., Partington, J.R.: Analysis of fractional delay systems of retarded and neutral type. Automatica **38**, 1133–1138 (2002)
25. Bonnet, C., Partington, J.R.: Stabilization of some fractional delay systems of neutral type. Automatica **43**, 2047–2053 (2007)
26. Liu, S., Zhou, X.F., Li, X., Jiang, W.: Asymptotical stability of Riemann–Liouville fractional singular systems with multiple time-varying delays. Appl. Math. Lett. **65**, 32–39 (2017)
27. Zhang, H., Ye, R., Cao, J., Alsaedi, A.: Delay-independent stability of Riemann–Liouville fractional neutral-type delayed neutral networks. Neural Process. Lett. **47**, 427–442 (2017)
28. Wen, L., Yu, Y., Wang, W.: Generalized Halanay inequalities for dissipativity of Volterra functional differential equations. J. Math. Anal. Appl. **347**, 169–178 (2008)
29. Stamova, I.M.: On the Lyapunov theory for functional differential equations of fractional order. Proc. Am. Math. Soc. **144**, 1581–1593 (2016)
30. Thuan, M.V., Huong, D.C.: New results on stabilization of fractional-order nonlinear systems via an LMI approach. Asian J. Control **20**, 1–10 (2018)
31. Zhang, G.D., Zeng, Z.G.: Exponential stability for a class of memristive neural networks with mixed time-varying delays. Appl. Math. Comput. **321**, 544–554 (2018)