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Backstepping-based observer for output feedback stabilization of a boundary controlled fractional reaction diffusion system

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Abstract— **This paper is concerned with observer-based output feedback boundary control for a fractional reaction diffusion (FRD) system. The considered FRD system is endowed with only boundary sensing available and boundary actuation. First, to design a backstepping-based observer for the FRD system. Second, to combine a separately backstepping-based feedback controller and the proposed observer to generate an output feedback controller for stabilizing the FRD system. Third, to analyze the Mittag-Leffler stability of the observer error system and the controlled FRD system. Finally, to verify the validness of our proposed method for the controlled FRD system and the observer error system through a numerical example.**

I. INTRODUCTION

Stabilization problem of fractional-order differential systems is an essential point in fractional-order control theory and has been studied by many researchers. In [1], stability results for control theory were provided for fractional-order differential systems in state-space form and polynomial representation. Due to the wide application of FRD systems in [2], these dynamic systems have got much attention. Recently, above corresponding efforts in [1] have been applied to the work of boundary feedback stabilization of such FRD systems by Ge et al. [3], with the aid of the backstepping technique [4].

Due to technical and economic restrictions, not all system state can be completely accessible or obtained. It induces the necessity of estimation of system state and observer-based control. The observer-based controller for fractional-order systems can be traced back to the work of Matignon et al. [5]. In this paper, our objective is to design the stabilizing output feedback controller by the observer and the backstepping technique, and finally obtain the Mittag-Leffler stability of the boundary controlled FRD system with Robin or mixed boundary conditions which is different from the one in [3]. To address this problem, we resort to the works of [6], which can be viewed as a breakthrough for development of observer design for distributed parameter systems (DPSs). In their work, backstepping observer and observer-based boundary feedback controller have been proposed for partial differential equations (PDEs). This design method is also suitable to FRD systems if the solution of stabilization problem has been obtained (see Section II-B).

Note that Matignon et al. [5] investigated observer-based stabilization problem of fractional-order differential systems and proposed the stability criterion of system matrix's argument. This criterion, however, is really hard to be used, since solving the argument of system matrix (system operator for fractional-order partial differential equations) is complex in fractional-order cases. Here, we discuss the observer-based stabilization problem of FRD systems by the fractional-order Lyapunov direct method, the Mittag-Leffler stability theory [7], and the novelty lemma [8] for Caputo fractional derivative which could find a simple Lyapunov candidate function for fractional-order systems.

The structure of this paper is considered as follows. In Section II, we introduce the problem statement and outline the backstepping-based boundary feedback control design. Section III and Section IV mainly focus on observer design and observer-based output boundary feedback control for the FRD system with Dirichlet actuation for anti-collocated and collocated cases. In Section V, a numerical simulation example has been used to test the efficiency of our proposed method. And finally, concluding remarks and future work are given in Section VI.

II. PROBLEM STATEMENT AND OVERVIEW ON BACKSTEPPING CONTROL DESIGN

A. Mathematical modeling

We consider the time FRD system [3], in this paper, whose state equation is given by

$$\begin{aligned} &{}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t) + a(x)u(x,t),\ x \in (0,1), t>0,\\ &u(x,0) = u_0(x), \quad x \in [0,1] \end{aligned} \tag{1}$$

with boundary conditions

$$p_1 u_x(0,t) - p_2 u(0,t) = 0, \quad t>0, \tag{2}$$

$$q_1 u_x(1,t) + q_2 u(1,t) = U(t), \quad t>0, \tag{3}$$

where $u(x,0) = u_0(x)$ represents the nonzero initial value, $p_1, p_2 > 0$, $q_1, q_2 \geq 0$, $U(t)$ is a control put, and ${}_0^C D_t^\alpha u(\cdot)$ denotes the Caputo time fractional derivative of α order [9]. It is shown in [1], [3] the FRD system (1)-(2), without control at $x=1$, i.e.,

$$q_1 u_x(1,t) + q_2 u(1,t) = 0, \quad t>0 \tag{4}$$

will be unstable if $a(\cdot)$ is a large positive function.

Based on the result of [1], [5], the sufficient and necessary condition for stability of the FRD system (1) with (2) and (4)

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is the roots of some polynomial lie outside the closed angular sector $|\arg(\text{spec}(\bar{\mathcal{A}}+a(x)))| \le \frac{\alpha\pi}{2}$, where the operator $\bar{\mathcal{A}}$ described by $(\bar{\mathcal{A}}u)(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}$. If $a(x)$ is large enough, the stability of FRD system (1) with (2) and (4) can be lost even if the eigenvalues of the operator $\bar{\mathcal{A}}$ are always negative. Thus, we consider to design backstepping-based output feedback boundary control to stabilize this system.

First, to design a Mittag-Leffler convergent observer for this system with only boundary measurement available and Dirichlet acutation at $x = 1$ for anti-collocated and collocated cases. Then, combining the designed observer and the backstepping-based boundary feedback controller in Section II-B, we obtain the output feedback boundary controller.

B. Overview of Dirichlet, Neumann and Robin boundary feedback control design

Our main topic of design observer and output feedback control recurs to the results of backstepping-based boundary feedback control. In this part, we will summarize it for the stabilization problem of the FRD system, which was solved in [10]. Let us consider the system (1) with boundary conditions

$$u_x(0,t) - pu(0,t) = 0, \tag{5}$$

$$u(1,t) = U(t), \tag{6}$$

$$\text{or } u_x(1,t) = U(t), \tag{7}$$

$$\text{or } u_x(1,t) + qu(1,t) = U(t), \tag{8}$$

where $t > 0$, $p = \frac{p_2}{p_1} > 0$ and $q = \frac{q_2}{q_1} > 0$. The backstepping approach's main idea is to utilize the integral transformation [11]

$$w(x,t) = u(x,t) + \int_0^x k(x,y)u(y,t)\mathrm{d}y \tag{9}$$

along with respective boundary feedback control to map the above system (1), (5)-(8) into the below target system

$${}_0^C D_t^\alpha w(x,t) = w_{xx}(x,t) - \lambda w(x,t), x \in (0,1), t > 0$$

with the initial condition

$$w(x,0) = w_0(x), \quad x \in [0,1]$$

and boundary conditions

$$w_x(0,t) - p^s w(0,t) = 0, \quad t > 0,$$
$$w(1,t) = 0,$$
$$\text{or } w_x(1,t) = 0,$$
$$\text{or } w_x(1,t) + q^s w_(1,t) = 0,$$

where $t > 0$, $\lambda > 0$, $p^s > 0$, $q^s > 0$ and $w_0(x) = u_0(x) + \int_0^x k(x,y)u_0(y)\mathrm{d}y$. Using Lemma 2.4 in [11] and Definition 3.1 in [7], we can easily obtain above integral transformation (9) is invertible and $u(x,t) = 0$ is an equilibrium point of the FRD system (1).

Based on the argument in [10, Section 3.1], we can easily show the below gain kernel PDE

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = \big(a(y)+\lambda\big)\, k(x,y), \\ \dfrac{\mathrm{d}}{\mathrm{d}x}k(x,x) = \dfrac{a(x)+\lambda}{2}, \\ k_y(x,0) = p\, k(x,0), \\ k(0,0) = p^s - p, \end{cases} \tag{10}$$

for $(x,y) \in \Xi = \{0 \le y \le x \le 1\}$.

Once the kernel gain $k(x,y)$ is determinate, Dirichlet, Neumann and Robin boundary feedback controllers at $x = 1$ can be expressed as follows

$$u(1,t) = -\int_0^1 k(1,y)u(y,t)\mathrm{d}y, \tag{11}$$

$$u_x(1,t) = -k(1,1)u(1,t) - \int_0^1 k_x(1,y)u(y,t)\mathrm{d}y, \tag{12}$$

$$\begin{aligned} u_x(1,t) + qu(1,t) =& \big(q - q^s - k(1,1)\big)u(1,t) \\ &- \int_0^1 \big(k_x(1,y) + q^s k(1,y)\big)u(y,t)\mathrm{d}y. \end{aligned} \tag{13}$$

respectively. Thus, it is easy for us to state the following theorems proved in [10].

Theorem 1. *It is assumed that $a(x) \in C^1[0,1]$. Thus, the gain kernel PDE* (10) *has a unique solution that is twice continuously differentiable in $0 \le y \le x \le 1$.*

Theorem 2. *Suppose that λ is an arbitrary positive constant, $a(\cdot)$ is an arbitrary function in $L^2(0,1)$ and $H^1(0,1)$, the function $w(\cdot,t)$ is continuously differentiable for $t \in [0,\infty)$, and the Laplace transform of $w^2(\cdot,t)$ exists. For any initial value $u_0(x) \in C(0,1)$ and $H^1(0,1)$ with compatible condition*

$$u_x(0,0) = pu_0(0), \quad u_0(1) = -\int_0^1 k(1,y)u_0(y)\mathrm{d}y, \tag{14}$$

$$\begin{aligned} \text{or } u_x(0,0) =& pu_0(0), \\ u_x(1,0) =& -k(1,1)u_0(1) - \int_0^1 k_x(1,y)u_0(y)\mathrm{d}y. \end{aligned} \tag{15}$$

$$\begin{aligned} \text{or } u_x(0,0) =& pu_0(0), \\ u_x(1,0) =& -qu(1,0) + \big(q - q^s - k(1,1)\big)u_0(1) \\ &- \int_0^1 \big(k_x(1,y) + q^s k(1,y)\big)u_0(y)\mathrm{d}y, \end{aligned} \tag{16}$$

the system (1) *with* (5) *and* (6)*, or with* (5) *and* (7)*, or with* (5) *and* (8) *has a unique solution under the above controllers* (11)*,* (12) *or* (13)*, and this system is L^2 and H^1 Mittag-Leffler stable at $u(x,t) = 0$.*

III. OBSERVER-BASED OUTPUT FEEDBACK CONTROL FOR ANTI-COLLOCATED CASE

As we know, for the observer-based output feedback control problem, much technical difference can not be found between Dirichlet, Neumann and Robin actuation. Without loss of generality, we will discuss observer design and observer-based output feedback control for anti-collocated case under Dirichlet actuation in this section.

A. Design of observer

First, let us consider the case of Dirichlet actuation at $x = 1$ when the measurement only available at $x = 0$. The corresponding observer is given by

$$\begin{aligned} {}_0^C D_t^\alpha \hat{u}(x,t) =& \hat{u}_{xx}(x,t) + a(x)\hat{u}(x,t) + r_1(x)(u(0,t) \\ &- \hat{u}(0,t)),\ x \in (0,1), \\ \hat{u}(x,0) =& \hat{u}_0(x), \quad x \in [0,1] \end{aligned} \tag{17}$$

with boundary conditions

$$\begin{aligned}&\hat{u}_x(0,t)=pu(0,t)+r_{10}(u(0,t)-\hat{u}(0,t)),\ t>0,\\&\hat{u}(1,t)=U(t),\ t>0,\end{aligned}\tag{18}$$

where $\hat{u}_0(x)$ is the initial value. Note that $r_1(x)$ is a observer gain function to be designed and r_{10} is a constant independent of x to be determined.

The observer error system can be given by

$$\begin{aligned}{}_0^C D_t^\alpha \tilde{u}(x,t)=&\tilde{u}_{xx}(x,t)+a(x)\tilde{u}(x,t)\\&-r_1(x)\tilde{u}(0,t),\ x\in(0,1)\\\tilde{u}(x,0)=&\tilde{u}_0(x),\quad x\in[0,1]\end{aligned}\tag{19}$$

with boundary conditions

$$\begin{aligned}&\tilde{u}_x(0,t)=-r_{10}\tilde{u}(0,t),\ t>0,\\&\tilde{u}(1,t)=0,\ t>0,\end{aligned}\tag{20}$$

where $\tilde{u}(x,t)=u(x,t)-\hat{u}(x,t)$, $\tilde{u}_0(x)$ is the initial value.

We need to choose appropriate observer gains $r_1(x)$ and r_{10} to stabilize system (19)-(20). For this purpose, we try to look for a integral transformation like

$$\tilde{u}(x,t)=\tilde{w}(x,t)+\int_0^x r(x,y)\tilde{w}(y,t)\mathrm{d}y\tag{21}$$

to convert the system (19)-(20) into a target system

$$\begin{aligned}&{}_0^C D_t^\alpha \tilde{w}(x,t)=\tilde{w}_{xx}(x,t)-\tilde{\lambda}\tilde{w}(x,t),\\&\tilde{w}(x,0)=\tilde{w}_0(x),\quad x\in[0,1]\end{aligned}\tag{22}$$

with boundary conditions

$$\begin{aligned}&\tilde{w}_x(0,t)=0,\ t>0,\\&\tilde{w}(1,t)=0,\ t>0,\end{aligned}\tag{23}$$

where $\tilde{\lambda}>0$ that is different from λ in Section II-A and can determine the observer convergence speed, $\tilde{w}_0(x)$ is the initial value and $\tilde{w}_0(x)=\tilde{u}_0(x)-\int_0^x r(x,y)\tilde{w}_0(y)\mathrm{d}y$. Obviously, the target error system is Mittag-Leffler stable in L^2 and H^1 (see the proof of Theorem 3 for more details).

Next, we will find out gain kernel $r(x,y)$ in (21) through some computation and substitution. First, taking the Caputo time fractional derivative of the integral transform (21) along the trajectory of the first equation in (22), then we get

$$\begin{aligned}{}_0^C D_t^\alpha \tilde{u}(x,t)=&{}_0^C D_t^\alpha \tilde{w}(x,t)+\int_0^x r(x,y){}_0^C D_t^\alpha \tilde{w}(y,t)\mathrm{d}y\\=&\tilde{w}_{xx}(x,t)-\tilde{\lambda}\tilde{w}(x,t)+\int_0^x r(x,y)(\tilde{w}_{yy}(y,t)\\&-\tilde{\lambda}\tilde{w}(y,t))\mathrm{d}y.\end{aligned}\tag{24}$$

First and second derivatives for (21) on x are given by

$$\begin{aligned}\tilde{u}_x(x,t)=&\tilde{w}_x(x,t)+r(x,x)\tilde{w}(x,t)\\&+\int_0^x r_x(x,y)\tilde{w}(y,t)\mathrm{d}y,\end{aligned}\tag{25}$$

$$\begin{aligned}\tilde{u}_{xx}(x,t)=&\tilde{w}_{xx}(x,t)+\frac{\mathrm{d}}{\mathrm{d}x}r(x,x)\tilde{w}(x,t)+r_x(x,x)\tilde{w}(x,t)\\&+\int_0^x r_{xx}(x,y)\tilde{w}(y,t)\mathrm{d}y.\end{aligned}\tag{26}$$

Note that $\frac{\mathrm{d}}{\mathrm{d}x}r(x,x)=r_x(x,x)+r_y(x,x)$, where $r_x(x,x)=\frac{\partial}{\partial x}r(x,y)|_{y=x}, r_y(x,x)=\frac{\partial}{\partial y}r(x,y)|_{y=x}$.

After a series of computation, we get the following form of kernel PDE

$$\begin{cases}r_{xx}(x,y)-r_{yy}(x,y)=-\big(a(x)+\tilde{\lambda}\big)\,r(x,y),\\\dfrac{\mathrm{d}}{\mathrm{d}x}r(x,x)=-\dfrac{1}{2}(a(x)+\tilde{\lambda}),\\r(1,y)=0,\end{cases}\tag{27}$$

for $(x,y)\in\Xi=\{0\le y\le x\le 1\}$.

And, it also yields

$$\begin{aligned}&r(x,0)\tilde{w}_x(0,t)-(r_1(x)+r_y(x,0))\tilde{w}(0,t)=0,\\&\tilde{w}_x(0,t)=(-r(0,0)-r_{10})\tilde{w}(0,t).\end{aligned}\tag{28}$$

Combining above equation (28) and (23), we get the following observer gains as long as the kernel PDE (27) has a unique solution

$$r_1(x)=-r_y(x,0),\quad r_{10}=-r(0,0).\tag{29}$$

Using the method of change of variables in [6, Section 3], we can transform the above kernel PDE (27) into the one (with $p=\infty$, $\tilde{\lambda}$ instead of λ) in class (10) in Section II-B. Note that the second equation in above kernel PDE (27) has contrary sign with the counterpart of the kernel PDE in [6, Section 3] since the mathematical sign in coordinate transformations is opposite. Using Theorem 1, we can obtain the kernel PDE (27) is well-posed.

Remark 1. In above kernel PDE (27), if $a(x)$ is a constant ν, using the above transformed kernel gain PDE (with $p=\infty$, $\tilde{\lambda}$ instead of λ) in class (10) in Section II-B and the conclusion in [12, page 35], we can obtain its solution $r(x,y)=(\nu+\tilde{\lambda})(1-x)\frac{I_1(\sqrt{(\nu+\tilde{\lambda})(2-x-y)(x-y)})}{\sqrt{(\nu+\tilde{\lambda})(2-x-y)(x-y)}}$. Moreover, based on (29), observer gains can be rewritten as $r_1(x)=-r_y(x,0)=\frac{(\nu+\tilde{\lambda})(1-x)}{x(2-x)}I_2(\sqrt{(\nu+\tilde{\lambda})x(2-x)})$, $r_{10}=-r(0,0)=-\frac{(\nu+\tilde{\lambda})}{2}$. They will be used for our numerical simulations in Section V. Note that $I_i(\cdot)$ denotes the modified Bessel functions with i order, $i=0,1,2$.

Based on the argument on invertibility of above integral transformation (9) provided in Section II-B, the integral transformation (21) is also invertible. Then we can get the following main result.

Theorem 3. *If $r(x,y)$ is the unique solution of kernel PDE* (27)*, then the observer error system* (19)-(20) *with observer gains $r_1(x)$ and r_{10} provided in* (29) *is Mittag-Leffler stable at $\tilde{u}(x,t)=0$ (equilibrium point of* (19)*) in $L^2(0,1)$ and $H^1(0,1)$ norms for any initial $\tilde{u}_0(x)\in L^2(0,1)$ and $H^1(0,1)$.*

Proof. Considering the Caputo time fractional derivatives of Lyapunov functional $V(t)=\frac{1}{2}\int_0^1\tilde{w}^2(x,t)\mathrm{d}x$ and $G(t)=\int_0^1\tilde{w}_x^2(x,t)\mathrm{d}x$ along with the target system (22)-(23) for L^2 and H^1 Mittag-Leffler stabilities respectively, we can get the target system (22)-(23) is L^2 and H^1 Mittag-Leffler stable. Then combining it and the invertibility of integral transformation (21), this conclusion can be obtained. □

B. Design of observer-based output feedback controller

In this part, we will combine the observer for Dirichlet actuation of anti-collocated case and the corresponding boundary feedback controller to address the output feedback control problem via the backstepping method. First, we illustrate the result from the following theorem.

Theorem 4. *If the $k(x,y)$ is the solution of* (10) *and $\tilde{\lambda}\ge\lambda$, the system* (1), (5), (6) *with the controller*

$$U(t)=-\int_0^1 k(1,y)\hat{u}(y,t)\mathrm{d}y\tag{30}$$

and the observer (17), (18), (30) *is L^2 and H^1 Mittag-Leffler stable at $u(x,t)=0$, $\hat{u}(x,t)=0$ (equilibrium point of* (17)*) for $u_0, \hat{u}_0 \in L^2(0,1)$ and $H^1(0,1)$, where observer gains $r_1(x)$ and r_{10} are provided in* (29).

Proof. Considering the integral transformation $\hat{w}(x,t) = \hat{u}(x,t)+\int_0^x k(x,y)\hat{u}(y,t)\mathrm{d}y$, we map the the observer (17), (18), (30) into the observer target system. This, together with the error target system (22)-(23), forms the below integrated $(\hat{w}(x,t),\tilde{w}(x,t))$ system

$$\begin{cases} {}_0^C D_t^\alpha \hat{w}(x,t) = \hat{w}_{xx}(x,t) - \lambda\hat{w}(x,t) \\ \qquad\qquad + \{r_1(x) + \int_0^x k(x,y)r_1(y)\mathrm{d}y\}\tilde{w}(0,t), \\ \hat{w}(x,0) = \hat{w}_0(x), \\ \hat{w}_x(0,t) = p^s\hat{w}(0,t) + (p+r_{10})\tilde{w}(0,t), \\ \hat{w}(1,t)=0, \\ {}_0^C D_t^\alpha \tilde{w}(x,t) = \tilde{w}_{xx}(x,t) - \tilde{\lambda}\tilde{w}(x,t), \\ \tilde{w}(x,0) = \tilde{w}_0(x), \\ \tilde{w}_x(0,t) = 0, \\ \tilde{w}(1,t) = 0, \end{cases} \tag{31}$$

where $\hat{w}_0(x)$ and $\tilde{w}_0(x)$ are initial values.

To show the Mittag-Leffler stability of the integrated $(\hat{w}(x,t),\tilde{w}(x,t))$ system, we consider the Lyapunov functional

$$V(t) = \frac{M}{2}\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x + \frac{1}{2}\int_0^1 \hat{w}^2(x,t)\mathrm{d}x, \tag{32}$$

where M is a positive constant to be set later.

By [8, Lemma 1] and integrating from 0 to 1 by parts, we can obtain the Caputo time fractional derivative of α order of $V(t)$ along the trajectory of (31) as follows

$$\begin{aligned} {}_0^C D_t^\alpha V(t) =& \frac{M}{2}\int_0^1 {}_0^C D_t^\alpha \tilde{w}^2(x,t)\mathrm{d}x \\ &+ \frac{1}{2}\int_0^1 {}_0^C D_t^\alpha \hat{w}^2(x,t)\mathrm{d}x \\ \le& M\int_0^1 \tilde{w}(x,t){}_0^C D_t^\alpha \tilde{w}(x,t)\mathrm{d}x \\ &+ \int_0^1 \hat{w}(x,t){}_0^C D_t^\alpha \hat{w}(x,t)\mathrm{d}x \\ =& -M\int_0^1 \tilde{w}_x^2(x,t)\mathrm{d}x - M\tilde{\lambda}\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x \\ &- p^s\hat{w}^2(0,t) - (p+r_{10})\hat{w}(0,t)\tilde{w}(0,t) \\ &- \int_0^1 \hat{w}_x^2(x,t)\mathrm{d}x - \lambda\int_0^1 \hat{w}^2(x,t)\mathrm{d}x \\ &+ \int_0^1 \tilde{w}(0,t)\hat{w}(x,t)\big(r_1(x) \\ &+ \int_0^x k(x,y)r_1(y)\mathrm{d}y\big)\mathrm{d}x. \end{aligned} \tag{33}$$

With the help of Poincaré and Young inequalities, the below estimate follows

$$\begin{aligned} -(p+r_{10})\hat{w}(0,t)\tilde{w}(0,t) \le& \frac{1}{4}\int_0^1 \hat{w}_x^2(x,t)\mathrm{d}x \\ &+ (p+r_{10})^2\int_0^1 \tilde{w}_x^2(x,t)\mathrm{d}x. \end{aligned} \tag{34}$$

Setting $N = \max_{0<x<1}\{r_1(x) + \int_0^x k(x,y)r_1(y)\mathrm{d}y\}$ and also using Poincaré and Young inequalities, we can get the following estimate

$$\begin{aligned} &\int_0^1 \tilde{w}(0,t)\hat{w}(x,t)\big(r_1(x) + \int_0^x k(x,y)r_1(y)\mathrm{d}y\big)\mathrm{d}x \\ &\le \tilde{w}(0,t)N\int_0^1 \hat{w}(x,t)\mathrm{d}x \\ &\le \frac{1}{4}\int_0^1 \hat{w}_x^2(x,t)\mathrm{d}x + 4N^2\int_0^1 \tilde{w}_x^2(x,t)\mathrm{d}x. \end{aligned} \tag{35}$$

These estimates (34), (35), together with (33), imply that

$$\begin{aligned} {}_0^C D_t^\alpha V(t) \le& -\left[M - (p+r_{10})^2 - 4N^2\right]\int_0^1 \tilde{w}_x^2(x,t)\mathrm{d}x \\ &- \frac{1}{2}\int_0^1 \hat{w}_x^2(x,t)\mathrm{d}x - M\tilde{\lambda}\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x \\ &- \lambda\int_0^1 \hat{w}^2(x,t)\mathrm{d}x. \end{aligned} \tag{36}$$

Since $\tilde{\lambda} \ge \lambda$, $\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x \le 4\int_0^1 \tilde{w}_x^2(x,t)\mathrm{d}x$ and $\int_0^1 \hat{w}^2(x,t)\mathrm{d}x \le 4\int_0^1 \hat{w}_x^2(x,t)\mathrm{d}x$, above estimate (36) can be further written as follows

$$\begin{aligned} {}_0^C D_t^\alpha V(t) \le& -\frac{1}{4}\left[M - (p+r_{10})^2 - 4N^2 + 4M\lambda\right] \\ &\times\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x - \frac{1}{8}(1+8\lambda)\int_0^1 \hat{w}^2(x,t)\mathrm{d}x. \end{aligned} \tag{37}$$

Moreover, we take the following equation of M

$$\frac{1}{4}\left[M - (p+r_{10})^2 - 4N^2 + 4M\lambda\right] = \frac{M}{8}(1+8\lambda),$$

which implies $M = 2(p+r_{10})^2 + 8N^2$.

Finally, choosing $M = 2(p+r_{10})^2 + 8N^2$, the equality (37) becomes

$${}_0^C D_t^\alpha V(t) \le -\frac{1}{4}(1+8\lambda)V(t). \tag{38}$$

Using the fractional-order extension of Lyapunov direction method [7, Theorem 5.1], we can obtain the integrated $(\hat{w}(x,t),\tilde{w}(x,t))$ system is L^2 Mittag-Leffler stable. In addition, since the $\tilde{w}(x,t)$ system and the $\hat{w}(x,t)$ system (without $\{r_1(x)+\int_0^x k(x,y)r_1(y)\mathrm{d}y\}\tilde{w}(0,t)$) are H^1 Mittag-Leffler stable, and the relationship between them is in series, then the integrated $(\hat{w}(x,t),\tilde{w}(x,t))$ system is also Mittag-Leffler stable in H^1. Therefore, the $(\hat{u}(x,t),\tilde{u}(x,t))$ system is Mittag-Leffler stable in L^2 and H^1, which indicates our result has been proved. □

Remark 2. These results on Theorem 3 and Theorem 4 can be generalized to Neumann actuation type with observer gains $r_1(x) = -r_y(x,0)$ and $r_{10} = -r(0,0)$, the controller $u_x(1,t) = -k(1,1)\hat{u}(1,t) - \int_0^1 k_x(1,y)\hat{u}(y,t)\mathrm{d}y$, and Robin actuation type with observer gains $r_1(x) = -r_y(x,0)$ and $r_{10} = -r(0,0)$, the controller $u_x(1,t) + qu(1,t) = (q - q^s - k(1,1))\hat{u}(1,t) - \int_0^1 \big(k_x(1,y) + q^s k(1,y)\big)\hat{u}(y,t)\mathrm{d}y$.

IV. OBSERVER-BASED OUTPUT FEEDBACK CONTROL FOR COLLOCATED CASE

In this part, we will solve the output feedback control problem with Dirichlet actuation for collocated case.

A. Observer design

Suppose measurement is only available at $x = 1$, which is at same end as Dirichlet actuation. Specifically, we focus on the observer problem with $u_x(1,t)$ measured and $u(1,t)$ actuated. Similarly, the corresponding observer is expressed as follows

$$\begin{aligned} {}_0^C D_t^\alpha \hat{u}(x,t) &= \hat{u}_{xx}(x,t) + a(x)\hat{u}(x,t) + r_1(x)(u_x(1,t) \\ &\quad - \hat{u}_x(1,t)),\ x \in (0,1), \\ \hat{u}(x,0) &= \hat{u}_0(x), \quad x \in [0,1] \end{aligned} \tag{39}$$

with boundary conditions

$$\begin{aligned} &\hat{u}_x(0,t) = p\hat{u}(0,t),\ t > 0, \\ &\hat{u}(1,t) = U(t) + r_{10}(u_x(1,t) - \hat{u}_x(1,t)),\ t > 0, \end{aligned} \tag{40}$$

where $\hat{u}_0(x)$ is the initial value. $r_1(x)$ is a observer gain function to be designed and r_{10} is a constant independent of x to be determined. In this collocated case, measurement $u_x(1,t)$ replaces $u(0,t)$, $r_1(x)$ is with $u_x(1,t) - \hat{u}_x(1,t)$ and r_{10} is injected in the boundary condition at $x = 1$.

The observer error system is described by

$$\begin{aligned} {}_0^C D_t^\alpha \tilde{u}(x,t) &= \tilde{u}_{xx}(x,t) + a(x)\tilde{u}(x,t) \\ &\quad - r_1(x)\tilde{u}_x(1,t),\ x \in (0,1), \\ \tilde{u}(x,0) &= \tilde{u}_0(x), \quad x \in [0,1] \end{aligned} \tag{41}$$

with boundary conditions

$$\begin{aligned} &\tilde{u}_x(0,t) = p\tilde{u}(0,t),\ t > 0, \\ &\tilde{u}(1,t) = -r_{10}\tilde{u}_x(1,t),\ t > 0, \end{aligned} \tag{42}$$

where $\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$, $\tilde{u}(x,0)$ is the initial value.

We use the integral transformation like

$$\tilde{u}(x,t) = \tilde{w}(x,t) + \int_x^1 r(x,y)\tilde{w}(y,t)\mathrm{d}y \tag{43}$$

to map the above system (41) and (42) into the below target system

$$\begin{aligned} &{}_0^C D_t^\alpha \tilde{w}(x,t) = \tilde{w}_{xx}(x,t) - \tilde{\lambda}\tilde{w}(x,t), \\ &\tilde{w}(x,0) = \tilde{w}_0(x), \quad x \in [0,1] \end{aligned} \tag{44}$$

with boundary conditions

$$\begin{aligned} &\tilde{w}_x(0,t) = p\tilde{w}(0,t),\ t > 0, \\ &\tilde{w}(1,t) = 0,\ t > 0, \end{aligned} \tag{45}$$

where $\tilde{\lambda} > 0$, $\tilde{w}_0(x)$ is the initial value satisfying (43).

The method of solving gain kernel $r(x,y)$ of (43) is same as the one in anti-collocated case apart from integral transformation (43) instead of (21). Then we can get the below conditions on the gain kernel $r(x,y)$ of (43)

$$\begin{cases} r_{xx}(x,y) - r_{yy}(x,y) = -\big(a(x) + \tilde{\lambda}\big)\, r(x,y), \\ \dfrac{\mathrm{d}}{\mathrm{d}x} r(x,x) = \dfrac{1}{2}(a(x) + \tilde{\lambda}), \\ r_x(0,y) = pr(0,y), \\ r(0,0) = 0, \end{cases} \tag{46}$$

for $(x,y) \in \Xi = \{0 \le y \le x \le 1\}$, and the observer gains

$$r_1(x) = -r(x,1), \quad r_{10} = 0. \tag{47}$$

Also utilizing the method of change of variables in [6, Section 4], we can transform the above kernel PDE (46) into the one (10) (with $k(0,0) = 0$) in Section II-B. Furthermore, using Theorem 1, we can obtain the kernel PDE (46) is well-posed. This, together with the invertibility of above integral transformation (43) provided in [6, Sectioin 4], induces the following main result.

Theorem 5. *If $r(x,y)$ is the unique solution of kernel PDE* (46)*, then the observer error system* (41)-(42) *with observer gains $r_1(x)$ and r_{10} provided in* (47) *is Mittag-Leffler stable at $\tilde{u}(x,t) = 0$ (equilibrium point of* (41)*) in $L^2(0,1)$ and $H^1(0,1)$ norms for any initial $\tilde{u}_0(x) \in L^2(0,1)$ and $H^1(0,1)$.*

B. Observer-based output feedback control law

A similar problem of observer-based output feedback controller design will hold for the collocated case and we will formulate our main result in below theorem.

Theorem 6. *If the $k(x,y)$ is the solution of* (10) *and $\tilde{\lambda} \ge \lambda$, the system* (1), (5), (6) *with the controller*

$$U(t) = -\int_0^1 k(1,y)\hat{u}(y,t)\mathrm{d}y \tag{48}$$

and the observer (39), (40) *(with $r_{10} = 0$),* (48) *is L^2 and H^1 Mittag-Leffler stable at $u(x,t) = 0$, $\hat{u}(x,t) = 0$ (equilibrium point of* (39)*) for $u_0, \hat{u}_0 \in L^2(0,1)$ and $H^1(0,1)$, where observer gains $r_1(x)$ and r_{10} are provided in* (47).

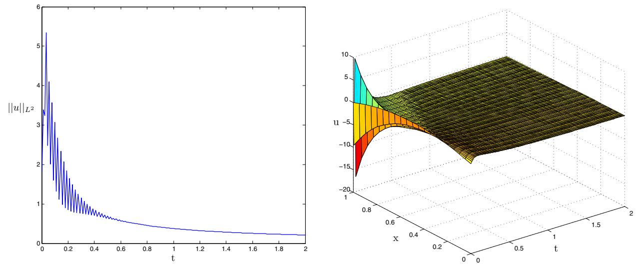
Proof. We omit the proof, since it is same as the one of Theorem 4 except that the transformation (21) is replaced by (43) and the $(\hat{u}(x,t), \tilde{u}(x,t))$ system is driven by $u_x(1,t) - \hat{u}_x(1,t)$ instead of $u(0,t) - \hat{u}(0,t)$. □

Remark 3. We also can extend these conclusions on Theorem 5 and Theorem 6 to Neumann actuation type with observer gains $r_1(x) = r_y(x,1), r_{10} = -r(1,1)$, and the controller $u_x(1,t) = -k(1,1)u(1,t) - \int_0^1 k_x(1,y)\hat{u}(y,t)\mathrm{d}y$.

V. NUMERICAL SIMULATION STUDY

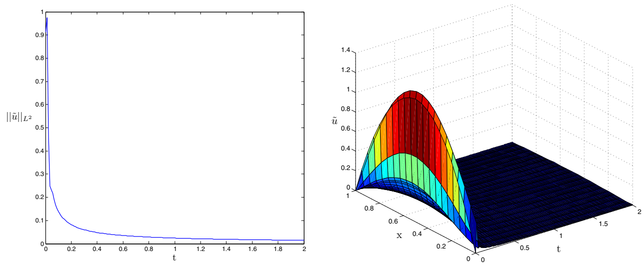
In this section, we will carry out a simulation example for the anti-collocated case to verify the effective of our results. We utilize the numerical algorithm in [13, Sectioin IV], the method of using difference to estimate differential, and finite-difference approximation method to simulate the FRD system. The spatial stepsize and temporal stepsize are $h = \frac{M}{X}, \nu = \frac{N}{T}$, where the spatial solution domain $x \in [0,M]$ and the temporal solution domain $t \in [0,N]$ with grid points $X+1$ and $T+1$ respectively.

In this anti-collocated case, the output $y(t)$ is $u(0,t)$, we set the discretization parameters $M = 1$, $N = 2$, $X = 20$, and $T = 180$. The initial conditions of the plant and observer are considered as $u_0(x) = 10x(1-x)$ and $\hat{u}_0(x) = 5x(1-x)$. The simulation parameters are given as $\alpha = 0.7$, $a(x) \equiv 10$, $p^s = p = 1$, $\tilde{\lambda} = 10$, and $\lambda = 1$. Then the observer gains for observer (17), (18), (30) and the gain kernel $k(x,y)$ [14, Theorem 10] for the controller (30) are given as $r_1(x) = \frac{20(1-x)}{x(2-x)} I_2(\sqrt{20x(2-x)})$, $r_{10} = -10$, $k(x,y) = -\frac{11}{2\sqrt{3}} \int_0^{x-y} e^{-\tau/2} I_0(\sqrt{11(x+y)(x-y-\tau)}) \sinh(\sqrt{3}\tau)\mathrm{d}\tau$ $+11x\frac{I_1(\sqrt{11(x^2-y^2)})}{\sqrt{11(x^2-y^2)}}$, where $k(0,0) = 0$.



(a) (b)

Fig. 1. Evolution of state L^2 norm and state of the system (1), (5) with the observer-based output feedback controller (6),(30).



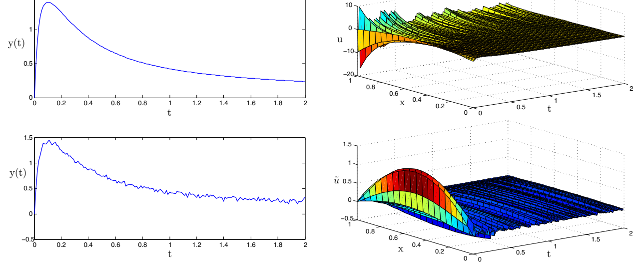
(a) (b)

Fig. 2. Evolution of state L^2 norm and state of the observer error system (19), (20).

The evolution of state L^2 norm and state of the system (1), (5) with the observer-based output feedback controller (6), (30) is shown in Fig. 1, which illustrates the controlled system is L^2 Mittag-Leffler stable (state norm converges to zero) and H^1 Mittag-Leffler stable (state converges to zero for all x). Fig. 2 shows the evolution of state L^2 norm and state of the observer error system (19), (20), which guarantees the L^2 and H^1 Mittag-Leffler stability of observer error system (19), (20). In Fig. 3 (a), the output $y(t) = u(0, t)$ in normal case and with white noise (signal-to-noise ratio 28 dB) are provided. Fig. 3 (b) implies the robust Mittag-Leffler stability of above controlled and observer error systems with measurement noise.

VI. Conclusions

This contribution considered observer design and observer-based output feedback boundary control for the FRD system with the Robin boundary condition at $x = 0$ and the observer-based output feedback boundary controller for Dirichlet actuation at $x = 1$ via the backstepping method. It is pointed out that the provided method here can be extended to more complicated families of systems, such as a class of coupled FRD systems, time delay FRD systems and FRD systems with disturbance, if the backstepping-based boundary feedback control problem of them can be solved. Future work will focus on these open equations which are still unsolved.



(a) (b)

Fig. 3. (a) Output $y(t)$ in normal case and with noise. (b) Evolution of state of the controlled and observer error systems with noise.

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