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# ON THE CONTROLLABILITY OF DISTRIBUTED-ORDER FRACTIONAL SYSTEMS WITH DISTRIBUTED DELAYS

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## ABSTRACT

*This paper investigates the controllability of distributed-order fractional systems with distributed delays. By using the controllability Gramian matrix and reduction to absurdity, a necessary and sufficient condition for the controllability of linear system is established, and a sufficient condition for the nonlinear system is obtained. Examples are given to illustrate the effectiveness of the theorems.*

## 1 INTRODUCTION

Fractional order systems are research hotspots recently. Fractional calculus appeared the same time as the appearance of calculus. Thanks to their nonlocal and heredity properties, they are widely used in a variety of fields such as control theory, biology, anomalous diffusion process, porous media,etc., these can be seen in the monographs [10, 12–14].

In the past few years, the controllability of fractional order systems have been widely researched, many results are obtained. In 2012, Wei Jiang [7] investigated a fractional control system with control delay and Balachandran [2] analyzed the controllability of fractional dynamical systems with distributed delays in control. In our recent work [5] and [6], we studied several kinds of fractional damped systems with different kinds of delays. In 2016, Joice Nirmala et al. [9] researched the controllability of nonlinear fractional delay dynamical systems using the fixed point argument. We can see that, delays are common phenomenons in actual model. There are many kinds of delays, such as fixed delays, time-varying delays and distributed delays. It is worth mention that, distributed delays are broad kind of delays, they can describe some special delays, this will be seem in equation (2) below.

For a given function $f(t)$, when we integrate ${}_0^C D_t^\alpha f(t)$ with respect to the order $\alpha$, then the distributed-order can be obtained. Distributed-order system is a generalization of fractional order system, it was first proposed by Caputo in 1969 [3]. Recently, much attention has been paid to the distributed-order systems and their applications on engineering fields. For some existing results of distributed-order, we refer readers to [1, 11, 15–17] and the reference therein.

In general, distributed-order fractional operator can be written as

$$ {}_0D_t^{q(\alpha)} f(t) := \int_{\gamma_1}^{\gamma_2} q(\alpha) {}_0^C D_t^\alpha f(t) d\alpha, $$

where $q(\alpha)$ denotes the weight function of the distributed order $\alpha \in [\gamma_1, \gamma_2]$.

In this paper, we investigates the following distributed-order

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fractional system with distributed delays

$$\begin{cases} \int_0^1 q(\alpha)_0^C D_t^\alpha x(t) d\alpha = \int_{-\tau}^0 d_\sigma B(t,\sigma) u(t+\sigma), t \geq 0, \\ x(0) = x_0, \\ u(t) = \psi(t), -\tau \leq t \leq 0, \end{cases} \tag{1}$$

where $0 < \alpha < 1, x \in \mathbb{R}^n$ is a state vector, $u(t) \in \mathbb{R}^m$ is a control vector, $\tau > 0$ is the time delay, and $\psi(t)$ is the initial control function. ${}_0^C D_t^\alpha x$ denotes the Caputo fractional derivative with respect to $x$ of order $\alpha$. We assume that meas$\{\alpha \in [0,1] | q(\alpha) > 0\} > 0$, which guarantees $\int_0^1 q(\alpha) s^\alpha d\alpha \neq 0$. In particular, if we define

$$B(t,\sigma) = \begin{cases} B, & \sigma = 0, \\ 0, & -\tau < \sigma < 0, \\ -C, & \sigma = -\tau, \end{cases}$$

and

$$q(\alpha) = \delta(\alpha - \beta), \ \ 0 < \beta < 1,$$

here $\delta$ is the Dirac delta function. By using the Lebesgue-Stieltjes integral and the property of Dirac delta function, system (1) becomes

$$\begin{cases} {}_0^C D_t^\beta x(t) = Bu(t) + Cu(t-\tau), \ \ 0 < \beta < 1, \ \ t \geq 0, \\ x(0) = x_0, \\ u(t) = \psi(t), -\tau \leq t \leq 0. \end{cases} \tag{2}$$

From this we can see that distributed-order system is a generalization of constant order system and distributed delay generalizes the constant delay.

**Definition 1.1.** *The left-side Caputo fractional derivative of order $\alpha > 0$ is defined by the operator*

$${}_0^C D_t^\alpha z(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} z^{(n)}(s) ds$$

*provided that it exists almost everywhere on $[0, +\infty)$ where $n = [\alpha] + 1$.*

**Definition 1.2.** *System (1) is said to be controllable on $J = [0,T]$, if, for each initial $x(0)$ and for each vector $x_1 \in \mathbb{R}^n$, there exists a control $u \in C(J)$ such that the corresponding solution of (1) with $x(0) = x_0$ satisfies $x(T) = x_1$.*

We proceed as follows. In Section 2, we present some lemmas that will be used in the discussion. Section 3 and Section 4 are the main results and their proofs. Examples are shown in Section 5 to verify the effective of the theorems.

## 2 PRELIMINARIES

In this section, we present several lemmas which will be used in the proof of the main results.

**Lemma 2.1.** *[8] The Laplace transform of $\int_0^1 q(\alpha)^C D_t^\alpha x(t) d\alpha$ is*

$$\begin{aligned} \mathscr{L}\{\int_0^1 q(\alpha)^C D_t^\alpha x(t) d\alpha\}(s) = X(s) \int_0^1 q(\alpha) s^\alpha d\alpha \\ -x(0) \frac{1}{s} \int_0^1 q(\alpha) s^\alpha d\alpha, \end{aligned} \tag{3}$$

*where $X(s) = \int_0^\infty x(t) e^{-st} dt$ is the Laplace transform of $x(t)$.*

**Lemma 2.2.** *The general solution of system*

$$\begin{cases} \int_0^1 q(\alpha)^C D_t^\alpha x(t) d\alpha = f(t), t \geq 0, \\ x(0) = x_0, \\ u(t) = \psi(t), -\tau \leq t \leq 0, \end{cases} \tag{4}$$

*with $0 < \alpha \leq 1$ can be written as*

$$x(t) = x_0 + \int_0^t (\mathscr{L}^{-1} \frac{1}{Q(\cdot)})(t-s) f(s) ds,$$

*where $Q(s) = \int_0^1 q(\alpha) s^\alpha d\alpha$ and $\mathscr{L}^{-1}$ is the inverse of Laplace transform.*

*Proof.* Take the Laplace transform on both side of equation (4), by Lemma 2.1, we get

$$Q(s)(X(s) - s^{-1} x_0) = F(s).$$

Then

$$X(s) = s^{-1} x_0 + Q^{-1}(s) F(s).$$

Taking inverse Laplace transform to both sides of the last expression, then

$$x(t) = x_0 + \int_0^t (\mathscr{L}^{-1} \frac{1}{Q(\cdot)})(t-s) f(s) ds.$$

According to Lemma 2.2, the solution of system (1) can be easily obtained.

**Lemma 2.3.** *The solution of system (1) can be written as*

$$x(t) = x_0 + \int_0^t (\mathscr{L}^{-1} \frac{1}{Q(\cdot)})(t-s) [\int_{-\tau}^0 d_\sigma B(s,\sigma) u(s+\sigma)] ds. \tag{5}$$

## 3 LINEAR SYSTEM

In this section, we investigate the controllability of the distributed-order fractional system (1) with distributed delays. First, we change the expression of the solution (5). For convenience, we denote $\mathscr{L}^{-1}Q^{-1}(\cdot)=P(\cdot)$.

$$\begin{aligned}
&x(t)\\
&=x_0+\textstyle\int_0^t P(t-s)\Big[\int_{-\tau}^0 d_\sigma B(s,\sigma)u(s+\sigma)\Big]ds\\
&=x_0+\textstyle\int_{-\tau}^0 dB_\sigma\Big[\int_0^t P(t-s)B(s,\sigma)u(s+\sigma)ds\Big]\\
&=x_0+\textstyle\int_{-\tau}^0 dB_\sigma\Big[\int_\sigma^0 P\big(t-(s-\sigma)\big)B(s-\sigma,\sigma)\psi(s)ds\Big]\\
&\quad+\int_{-\tau}^0 dB_\sigma\Big[\int_0^{t+\sigma} P\big(t-(s-\sigma)\big)B(s-\sigma,\sigma)u(s)ds\Big]\\
&=x_0+\textstyle\int_{-\tau}^0 dB_\sigma\Big[\int_\sigma^0 P\big(t-(s-\sigma)\big)B(s-\sigma,\sigma)\psi(s)ds\Big]\\
&\quad+\int_0^t\Big[\int_{-\tau}^0 P\big(t-(s-\sigma)\big)d_\sigma B_t(s-\sigma,\sigma)u(s)ds\Big],
\end{aligned} \tag{6}$$

where

$$B_t(s,\tau)=\begin{cases}B(s,\tau), & s\le t,\\ 0, & s>t,\end{cases} \tag{7}$$

and $dB_\sigma$ denotes the Lebesgue-Stieltjes integration with respect to the variable $\sigma$ in the function $B(t,\sigma)$.

Now, we introduce the notation

$$G(t,s)=\int_{-\tau}^0 P(t-(s-\sigma))d_\sigma B_t(s-\sigma,\sigma), \tag{8}$$

and for $T>0$, we define the controllability Gramian matrix

$$W=\int_0^T G(T,s)G^\top(T,s)ds. \tag{9}$$

**Theorem 3.1.** *The linear distributed-order fractional system (1) with distributed delays is controllable on J if and only if the controllability Gramian matrix*

$$W=\int_0^T G(T,s)G^\top(T,s)ds$$

*is positive defined.*

*Proof.* First, we suppose that $W$ is not positive definite. By the definition of $W$, we know that $W$ is not invertible, so there exists a nonzero $y$ such that

$$y^\top Wy=y^\top\int_0^T G(T,s)G^\top(T,s)dsy=0,$$

hence, for $s\in[0,T]$,

$$\begin{aligned}
&y^\top G(T,s)\\
&=y^\top\int_{-\tau}^0 P(T-(s-\sigma))^{\alpha-1}d_\sigma B_T(s-\sigma,\sigma)\\
&=0.
\end{aligned} \tag{10}$$

Since system (1) is controllable, there exists a control $u\in C(J)$ such that it steers the initial state $x_0=y$ to the origin in the interval $J$, we choose $\psi(t)=0$, then it follows that

$$\begin{aligned}
x(T)&=x_0+\int_0^T\Big[\int_{-\tau}^0 P(T-(s-\sigma))d_\sigma B_T(s-\sigma,\sigma)\Big]u(s)ds\\
&=y+\int_0^T\Big[\int_{-\tau}^0 P(T-(s-\sigma))d_\sigma B_T(s-\sigma,\sigma)\Big]u(s)ds\\
&=0,
\end{aligned}$$

from this, we get

$$y^\top y+\int_0^T y^\top G(T,s)u(s)ds=0. \tag{11}$$

Combine (10) and (11) , we have $y^\top y=0$. This is a contradiction to the assumption $y\neq 0$. Thus $W$ is invertible, hence, $W$ is positive defined.

Next, we suppose that $W$ is positive definite, then its inverse is well-defined. Define the control function as

$$\begin{aligned}
u(t)=\ &G^\top(T,t)W^{-1}\Bigg(x_1-x_0-\int_{-\tau}^0 dB_\sigma\\
&\times\Big(\int_\sigma^0 P(T-(s-\sigma)))B(s-\sigma,\sigma)\psi(s)ds\Bigg),
\end{aligned}$$

where $x_0$, $\psi(t)$ and $x_1$ are chosen arbitrarily. Substituting $u(t)$ into (6) and using the definition of $W$, we have

$$\begin{aligned}
&x(T)\\
&=x_0+\int_{-\tau}^0 dB_\sigma\Big(\int_\sigma^0 P(T-(s-\sigma))B(s-\sigma,\sigma)\psi(s)ds\Big)\\
&\quad+\int_0^T\Bigg[\int_{-\tau}^0 P(T-(s-\sigma))d_\sigma B_T(s-\sigma,\sigma)\\
&\qquad\times G^\top(T,s)W^{-1}\Bigg(x_1-x_0-\int_{-\tau}^0 dB_\sigma\\
&\qquad\times\int_\sigma^0 P(T-(s-\sigma))B(s-\sigma,\sigma)\psi(s)ds\Bigg)\Bigg]\\
&=x_1.
\end{aligned} \tag{12}$$

This means that the control $u(t)$ steers the initial $x_0$ to the desired vector $x_1\in\mathbb{R}^n$ at time $T$. Hence system (1) is controllable.

From Theorem 3.1, we have derived that the controllability of the linear distributed-order fractional system (1) with distributed delays is equivalent to the invertibility of $W$.

By Theorem 3.1, we have a simple pattern for the special case (2).

**Corollary 3.1.** *The fractional dynamical system with control delay (2) is controllable if and only if*

$$\mathrm{rank}[B,C] = n. \tag{13}$$

For system (2), the invertible of W is equivalent to the relation (13), this is a special case, and is easy to check.

## 4 NONLINEAR SYSTEM

In this section, we consider the following nonlinear distributed-order fractional system with distributed delays:

$$\begin{cases} \int_0^1 q(\alpha)^C D_{0+}^{\alpha} x(t) d\alpha = \int_{-\tau}^0 d_\sigma B(t,\sigma)u(t+\sigma) + f(t,x,u), t \geq 0, \\ x(0) = x_0, \\ u(t) = \psi(t), -\tau \leq t \leq 0, \end{cases} \tag{14}$$

where $A$ and $B$ are the same as above and $f : J \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function.

Denote

$$M = \{(z,v) : z \in C_n(J), v \in C_m(J)\},$$

with the uniform norm $\|(z,v)\| = \|z\| + \|v\| = \max_{t\in J} |z(t)| + \max_{t\in J} |v(t)|$, where $C_n(J) = \{f : J \to \mathbb{R}^n | f \text{ is continuous on } J\}$ is a Banach space, $|\cdot|_n$ and $|\cdot|_m$ denotes the max norm in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, without confusion, we use the notation $|\cdot|$. For each $(z,v) \in M$, consider the nonlinear distributed-order fractional system described by

$$\begin{cases} \int_0^1 q(\alpha)^C D_{0+}^{\alpha} x(t) d\alpha = \int_{-\tau}^0 d_\sigma B(t,\sigma)u(t+\sigma) + f(t,z,v), t \geq 0, \\ x(0) = x_0, \\ u(t) = \psi(t), -\tau \leq t \leq 0. \end{cases} \tag{15}$$

By Lemma 2.2 and (6), the solution of (15) can be written as

$$\begin{aligned} x(t) \ &= x_0 + \int_0^t P(t-s)f(s,z,v)ds \\ &+ \int_0^t [\int_{-\tau}^0 P(t-(s-\sigma))d_\sigma B_t(s-\sigma,\sigma)u(s)ds] \\ &+ \int_{-\tau}^0 dB_\sigma[\int_\sigma^0 P(t-(s-\sigma))B(s-\sigma,\sigma)\psi(s)ds], \end{aligned}$$

where $B_t(s,\tau)$ is given by (7), and $d_\sigma B_t$ denotes the Lebesgue-Stieltjes integration with respect to the variable $\sigma$ in the function $B_t(s-\sigma,\sigma)$. Brevity, for given $x_0, x_1 \in \mathbb{R}^n$, we use the notation

$$\begin{aligned} &\eta(x_0,x_1;z,v) \\ &= x_1 - x_0 - \int_0^T P(T-s)f(s,z,v)ds \\ &\quad - \int_{-\tau}^0 dB_\sigma \left[ \int_\sigma^0 P(T-(s-\sigma))B(s-\sigma,\sigma)\psi(s)ds \right], \end{aligned}$$

and define the control function

$$u(t) = G^\top(T,t)W^{-1}\eta(x_0,x_1;z,v),$$

where the initial state $x_0, \psi(t)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily.

In order to derive the controllability result of nonlinear distributed-order system, we need the following lemma.

**Lemma 4.1.** *[4] If the function $f(t,\nu)$ is bounded locally in $\nu$ and satisfies that*

$$\lim_{|\nu|\to\infty} \frac{|f(t,\nu)|}{|\nu|} = 0$$

*uniformly in $t \in J$. Then for every pair of constants $c,d$, there is a constant $r$ such that if $|\nu| \leq r$, then $c|f(t,\nu)| + d \leq r$ for all $t \in J$.*

**Theorem 4.1.** *Suppose that the continuous function $f$ satisfies the condition*

$$\lim_{|(x,u)|\to\infty} \frac{|f(t,x,u)|}{|(x,u)|} = 0 \tag{16}$$

*uniformly in $t \in J$, and the linear fractional system (1) is controllable. Then nonlinear system (14) is controllable on $J$.*

*Proof.* Since the system (1) is controllable, it follows from Theorem 3.1 that $W$ defined in (9) is invertible. We define the operator $\Psi : \ M \to M$ as follows:

$$\Psi(z,v) = (x,u),$$

where

$$\begin{aligned} u(t) &= G^\top(T,t)W^{-1}(\eta(x_0,x_1;z,v)) \\ &= G^\top(T,t)W^{-1}\Big(x_1 - x_0 - \int_0^T P(T-s)f(s,z,v)ds \\ &\quad - \int_{-\tau}^0 dB_\sigma[\int_\sigma^0 P(T-(s-\sigma))B(s-\sigma,\sigma)\psi(s)ds]\Big), \end{aligned} \tag{17}$$

and

$$\begin{aligned}
x(t) = x_0 + &\int_0^t P(t-s)f(s,z,v)ds \\
&+ \int_{-\tau}^0 d_\sigma B(s-\sigma,\sigma)\big[\int_\sigma^0 P(t-(s-\sigma))\psi(s)ds\big] \\
&+ \int_0^t \big[\int_{-\tau}^0 P(t-(s-\sigma))d_\sigma B(s-\sigma,\sigma)\big]u(s)ds.
\end{aligned} \tag{18}$$

Now, we show that there exists a constant $r>0$ such that

$$\Psi(M(r)) \subset M(r) \text{ holds,}$$

where $M(r) = \{(z,v) \in M : \|z\| \leq \frac{r}{2} \text{ and } \|v\| \leq \frac{r}{2}\}$. For convenience, we introduce the following constants:

$$\begin{aligned}
&a_1 = \sup_{s\in[0,T]} \|P(T-s)\|, \\
&a_2 = \| \int_{-\tau}^0 d_\sigma B(s-\sigma,\sigma)[\int_\sigma^0 P(t-(s-\sigma))\psi(s)ds]\|, \\
&a_3 \sup_{t\in[0,T]} \|G^\top(T,t)\|, \\
&a = \max_{s\in[0,T]}\{T\|G(T,s)\|,1\}, \\
&c_1 = 4a_1a_3T|W^{-1}, \qquad c_2 = 4a_1T, \\
&d_1 = 4a_3|W^{-1}|(|x_1|+|x_0|+a_2), \qquad d_2 = 4(x_0+a_2), \\
&c = \max\{ac_1,c_2\}, \qquad d = \max\{ad_1,d_2\}, \\
&\sup|f| = \sup\{|f(s,z,v)|; s\in J\}.
\end{aligned}$$

By (18) and (17), we have

$$\begin{aligned}
&|u(t)| \\
&\leq \|G^\top(T,t)\||W^{-1}|\big[|x_1|+|x_0|+a_2+a_1T\sup|f|\big] \\
&\leq a_3|W^{-1}|[|x_1|+|x_0|+a_2]+a_3|W^{-1}|a_1T\sup|f| \\
&\leq \frac{d_1}{4}+\frac{c_1}{4}\sup|f|,
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
&|x(t)| \\
&\leq x_0+a_2+\int_0^T |G(t,s)||u(s)|ds + a_1T\sup|f| \\
&\leq \frac{d_2}{4}+a\left[\frac{d_1}{4}+\frac{c_1}{4}\sup|f|\right]+c_2\sup|f| \\
&\leq \frac{d}{2}+\frac{c}{2}\sup|f|.
\end{aligned} \tag{20}$$

Since $f(t,x,u)$ satisfies (16), by Lemma 4.1, for each pair of positive constants $c$ and $d$, there exists a positive constant $r$ such that, if $\|(\bar{z},\bar{v})\| \leq r$, then

$$c|f(t,\bar{z},\bar{v})|+d \leq r, \text{ for all } t\in[0,T]. \tag{21}$$

Now we take $c,d$ as given by (19), and choose $r$ such that (21) holds. Therefore, if $\|z\| \leq \frac{r}{2}$, and $\|v\| \leq \frac{r}{2}$, then $|z(s)|+|v(s)| \leq r$ for all $s\in[0,T]$, it follows that $d+c\sup|f| \leq r$. Therefore, by (19), we have $|u(s)| \leq \frac{r}{4}$ for all $s\in[0,T]$, and hence $\|u\| \leq \frac{r}{2}$, and by (20), $\|x\| \leq \frac{r}{2}$. Thus, $\Psi(M(r)) \subset M(r)$.

Similar to the proof of our previous paper [6], we get that $\Psi$ admits a fixed point $(z,v)\in M(r)$ such that $\Psi(z,v) = (z,v) \equiv (x,u)$. Hence $x(t)$ is the solution of system (14), and it is easy to verify that $x(T) = x_1$ and that the control function $u(t)$ steers system (14) from initial $x_0$ to $x_1$ on $[0,T]$. Hence system (14) is controllable on $[0,T]$.

## 5 EXAMPLES

In this section, we give examples to illustrate the effective of the theorems.

**Example 5.1.** *Consider the linear distributed-order system with distributed delays*

$$\int_0^1 \delta(\alpha-0.5)^C D_t^\alpha x(t)d\alpha = \int_{-1}^0 d_\sigma B(t,\sigma)u(t+\sigma), \tag{22}$$

*where*

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \text{ and } B(t,\sigma) = \begin{pmatrix} -\cos(t+\sigma) & \sin(t+\sigma) \\ -\sin(t+\sigma) & -\cos(t+\sigma) \end{pmatrix}.$$

*Then, by the properties of Dirac delta function, system (22) becomes*

$${}^C D_t^{0.5} x(t) = \int_{-1}^0 d_\sigma B(t,\sigma)u(t+\sigma).$$

*By calculation,* $P(t-(s-\sigma)) = \frac{(t-(s-\sigma))^{-0.5}}{\Gamma(0.5)}$ *and*

$$\begin{aligned}
G(T,s) &= \int_{-1}^0 \frac{(T-(s-\sigma))^{-0.5}}{\Gamma(0.5)} \begin{pmatrix} \sin(T+\sigma) & \cos(T+\sigma) \\ -\cos(T+\sigma) & \sin(T+\sigma) \end{pmatrix} d\sigma \\
&= \begin{pmatrix} p(s) & w(s) \\ -w(s) & p(s) \end{pmatrix},
\end{aligned}$$

*where*

$$p(s) = \int_{-1}^0 \frac{(T-(s-\sigma))^{-0.5}}{\Gamma(0.5)} \sin(T+\sigma)d\sigma,$$

$$w(s) = \int_{-1}^{0} \frac{(T-(s-\sigma))^{-0.5}}{\Gamma(0.5)} \cos(T+\sigma) d\sigma.$$

*Then, we obtain that the controllability matrix*

$$\begin{aligned} W &= \int_0^T G(T,s) G^\top(T,s) ds \\ &= \int_0^T (p^2(s) + w^2(s)) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ds \end{aligned}$$

*is positive defined for any $T > 0$. Therefore, by Theorem 3.1, system (22) is controllable.*

**Example 5.2.** *Now we consider the following distributed-order fractional system:*

$$\int_0^1 (\alpha - 1)^C D_t^\alpha x(t) d\alpha = \int_{-1}^{0} d_\sigma B(t,\sigma) u(t+\sigma), \tag{23}$$

*where $B(t,\sigma)$ equals to Example 5.1. Then we have $P(s) = \mathscr{L}^{-1} \frac{\ln^2 s}{1-s-\ln s}$,*

$$\begin{aligned} G(T,s) &= \int_{-1}^{0} \mathscr{L}^{-1} \Big( \frac{\ln^2(T-(s-\sigma))}{1-(T-(s-\sigma)) + \ln(T-(s-\sigma))} \Big) \\ &\quad \times \begin{pmatrix} \sin(T+\sigma) & \cos(T+\sigma) \\ -\cos(T+\sigma) & \sin(T+\sigma) \end{pmatrix} d\sigma \\ &= \begin{pmatrix} q(s) & v(s) \\ -v(s) & q(s) \end{pmatrix}, \end{aligned}$$

*where*

$$\begin{aligned} q(s) = \int_{-1}^{0} \mathscr{L}^{-1} \Big( \frac{\ln^2(T-(s-\sigma))}{1-(T-(s-\sigma)) + \ln(T-(s-\sigma))} \Big) \\ \times \sin(T+\sigma) d\sigma, \end{aligned}$$

$$\begin{aligned} v(s) = \int_{-1}^{0} \mathscr{L}^{-1} \Big( \frac{\ln^2(T-(s-\sigma))}{1-(T-(s-\sigma)) + \ln(T-(s-\sigma))} \Big) \\ \times \cos(T+\sigma) d\sigma. \end{aligned}$$

*Then, we obtain the controllability matrix*

$$\begin{aligned} W &= \int_0^T G(T,s) G^\top(T,s) ds \\ &= \int_0^T (q^2(s) + v^2(s)) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ds \end{aligned}$$

*is positive defined for any $T > 0$. Therefore, by Theorem 3.1, the system (23) is controllable.*

**Example 5.3.** *Base on Example 5.1 and Example 5.2, if we choose*

$$f(x) = \left( \frac{x_1 + x_2}{1 + x_1^2 + x_2^2}, \frac{x_1 + 1}{1 + x_1^2 + x_2^2} \right)^\top,$$

*then the corresponding nonlinear systems are controllable.*

## 6 Conclusions

In this paper, we have investigated the distributed-order fractional systems with distributed delays. First, we present the expression of the solution, then by using the controllability Gramian matrix, a necessary and sufficient condition of linear system is obtain, also, a special rank condition is derived. Third, utilize a strong restriction of $f$, we get the nonlinear result. Examples are given at last.

The methods we used here are the same as [2] and [6], but the distributed-order fractional systems are more general.


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