Proceedings of the ASME 2017 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference
IDETC/CIE 2017
August 6-9, 2017, Cleveland, Ohio, USA

DETC2017-67903

FURTHER REMARKS ON THE EXISTENCE OF PERIODIC SOLUTIONS OF LINEAR TIME VARYING PERIODIC FRACTIONAL ORDER SYSTEMS

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ABSTRACT

Existence of periodic solutions of fractional order dynamic systems is an important and difficult issue in fractional order systems field. In this paper, the non existence of completely periodic solutions and existence of partly periodic solutions of fractional order linear time varying periodic systems and fractional order nonlinear time varying periodic systems are discussed. A new property of Laplace transform of periodic function is derived. The non existences of completely periodic solutions of fractional order linear time varying periodic systems and fractional order nonlinear time varying periodic fractional order systems are presented by Laplace transform method and contradiction approach. The existence of partly periodic solutions of fractional order dynamic systems are proved by constructing numerical examples and considering Laplace transform property approaches. The examples and state figures are given to illustrate the effectiveness of conclusion presented.

1 INTRODUCTION

Fractional calculus is a mathematical topic of more than 300 years old, but its application to physics and engineering has been reported only in recent years. It has been found that in many practical cases, systems can be more adequately described by the fractional order differential equations. Nowadays, fractional-order dynamics, which are defined based on fractional-order differential equations (Podlubny, 1999), play a significant role in different control applications. In recent years, fractional calculus has attracted the attention of researchers in many fields such as engineering, biology, economics etc. One of the areas of interest in this field is existence of period solutions of the dynamical systems and chaos character of solutions which is investigated by many researchers

Due to the growing interest of fractional-order dynamics to be applied in different control applications, it seems analysis of this type of system is of great importance. This problem, i.e. analysis of fractional-order systems, is a motivation for some recent research works. In one of these works, the non-existence of periodic solutions in a class of fractional-order systems defined based on Caputo definition has been proved. By using the proof given by the property of Laplace transform a remarkable property for fractional-order derivatives of periodic functions is presented in this paper.

Thus for chaos control in fractional order systems, it is important to show that periodic solutions exist in such systems. Tavazoei and Haeri (2009) have proved that in these systems, periodic solution cannot be detected under any circumstances. In this paper, it is proved that this is not a general claim and periodic solution can be detected by considering some conditions.

The time varying fractional order systems presented in this paper does not have any so called completely periodic solution. It has been shown that by this assumption, harmonic balance method can be applied to fractional order systems. In the past

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ten years, the study of dynamical behavior of fractional order systems has attracted increasing attention. In this paper, a basic and simple proof approach of nonexistence completely periodic solution for dynamic fractional order systems is given by a new property of Laplace transform. The examples and state figures are given to illustrate the methods presented.

2 PRELIMINARIES

Let us denote by $\mathbf{Z}^+$ the set of positive integer numbers, denote by $\mathbf{C}$ the set of complex numbers, denote by $\mathbf{R}^{n\times n}$ the set of $n\times n$ dimension real numbers .We denote the real part of complex number α by $\mathrm{Re}(\alpha)$.

Caputo derivative has been often used in fractional order systems since it has the practical initial states like that of integer order systems.

Definition 1. The Caputo derivative of fractional order α of function $x(t)$ is defined as

$$ {}_0^C D_t^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} x^{(n)}(\tau) d\tau, $$

where $n-1<\alpha<n\in\mathbf{Z}^+$.

Definition 2. The Riemann-Liouvile derivative of fractional order α of function $x(t)$ is defined as

$$ {}_0^{RL} D_t^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-\tau)^{n-\alpha-1} x(\tau) d\tau, $$

where $n-1<\alpha<n\in\mathbf{Z}^+$.

Definition 3. The Grunwald-Letnikov derivative of fractional order α of function $x(t)$ is defined as

$$ {}_0^{GL} D_t^{\alpha} x(t) = \lim_{h\to 0} h^{-\alpha} \sum_{r=0}^{[(t-\alpha)/h]} (-1)^r C_{\alpha}^r x(t-rh), $$

where $n-1<\alpha<n\in\mathbf{Z}^+$.

Definition 4. The Mittag-Leffler function is defined as

$$ E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+1)}, $$

where $\mathrm{Re}(\alpha)>0, t\in\mathbf{C}$. The two-parameter Mittag-Leffler function is defined as

$$ E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+\beta)}, $$

where $\mathrm{Re}(\alpha)>0, \beta, t\in\mathbf{C}$.

Property 1. The Laplace transform of Caputo derivative $x(t)$ is

$$ \mathbf{L}({}_0^C D_t^{\alpha} x(t)) = s^{\alpha} X(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} x^k(0), $$

where $X(s)=\mathbf{L}[x](s), n-1<\alpha<n\in\mathbf{Z}^+$.

Property 2. Let $\alpha\in(0,\infty)\setminus\mathbf{N}$. Then, we have

$$ {}_0^{RL} D_t^{\alpha} x(t) = {}_0^{GL} D_t^{\alpha} x(t) = {}_0^{C} D_t^{\alpha} x(t) + \sum_{i=0}^{n-1} \frac{x^{(i)}(0)}{\Gamma(i-\alpha+1)} t^{k-\alpha}, $$

where $n-1<\alpha<n\in\mathbf{Z}^+$.

Lemma 1 The Laplace transform of $A\cos(\omega t)$ is:

$$ \mathbf{L}(A\cos(\omega t)) = \frac{As}{s^2+\omega^2}. $$

The Laplace transform of $A\sin(\omega t)$ is:

$$ \mathbf{L}(A\sin(\omega t)) = \frac{A\omega}{s^2+\omega^2}. $$

Lemma 2 The Laplace transform of $t^{\beta-1}E_{\alpha,\beta}(-\omega t^{\alpha})$ is:

$$ \mathbf{L}(t^{\beta-1}E_{\alpha,\beta}(-\omega t^{\alpha})) = \frac{s^{\alpha-\beta}}{s^{\alpha}+\omega}. $$

Lemma 3 The Laplace transform of n order derivative $f^n(t)$ is:

$$ \mathbf{L}(f^{(n)}(t)) = s^n F(s) - \sum_{i=0}^{n-1} s^{n-1-i} f^{(i)}(0). $$

3 PROPERTY OF LAPLACE TRANSFORMS FOR PERIODIC FUNCTION

For $T-$periodic function $x(t+T)=x(t)$, from

$$ \frac{d}{dt}x(t+T) = \frac{d}{d(t+T)}x(t+T)\frac{d}{dt}(t+T) = x'(t+T), $$

it is easy to see that $x^{(k)}(t+T)=x^{(k)}(t)$. From Fig.1, we can see that the state curves of fractional order systems are completely different from that of integer order systems. Integer order systems can exist periodic solution. Can fractional order systems also exist periodic solution? In the paper, the answer is negative, i.e. fractional order systems can not exist any periodic solution at all.

From Lemma 3, it is easy to verify the following property.

Theorem 1 The Laplace transform of a $T-$periodic function $x(t)$ is given as

$$\mathbf{L}(x(t)) = \frac{\int_0^T e^{-s\tau}x(\tau)d\tau}{1-e^{-sT}}.$$

proof By taking Laplace transform for $x(t)$, we have

$$\mathbf{L}(x(t)) = \int_0^{\infty} e^{-st}x(t)dt$$

$$= \sum_{i=0}^{\infty}\int_{iT}^{(i+1)T} e^{-st}x(t)dt$$

$$= \sum_{i=0}^{\infty}\int_0^T e^{-s(\tau+iT)}x(\tau)d\tau$$

$$= \sum_{i=0}^{\infty} e^{-isT}\int_0^T e^{-s\tau}x(\tau)d\tau$$

$$= \frac{\int_0^T e^{-s\tau}x(\tau)d\tau}{1-e^{-sT}}.$$

Property 3. The Laplace transform of a $T-$periodic function $x(t)$ must be expressed as a integer power items series of s. i.e.

$$\mathbf{L}(x(t)) = \sum_{i=0}^{\infty} f_i(T)s^i.$$

Remark 1. Property 3 is only a necessary condition but not a sufficient condition. To illustrate this, we can give the following examples.

Example 1 For periodic function $f(t) = t, 0 \le t < 1, f(t+1) = f(t)$, it is easy to obtain its Laplace transform as follows.

$$\mathbf{L}(f(t)) = \int_0^{\infty} e^{-ts}f(t)dt = \sum_{i=0}^{\infty}\int_i^{i+1} e^{-ts}tdt$$

$$= \sum_{i=0}^{\infty}\int_i^{i+1} e^{-ts}tdt$$

$$= \sum_{i=0}^{\infty} e^{si}\int_0^1 e^{-ts}tdt$$

$$= \sum_{i=0}^{\infty} e^{si}(\frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}),$$

$$= s^{-2} + s^{-1} + 1 + 2s^2 + \cdots.$$

It can be seen that Laplace transform of periodic function $f(t) = t$ can be expressed as an integer power series of s. And Laplace transforms of periodic function $\cos(t)$ and $\sin(t)$ can also be expressed as the integer power series of s.

$$\mathbf{L}(\cos(t)) = \frac{s}{s^2+1} = \sum_{i=0}^{\infty}(-1)^i s^{2i+1},$$

$$\mathbf{L}(\sin(t)) = \frac{1}{s^2+1} = \sum_{i=0}^{\infty}(-1)^i s^{2i}.$$

But for aperiodic function $g(t) = t, t > 0$, it is easy to see that $\mathbf{L}(g(t)) = \frac{1}{s^2}$ also can be expressed as the integer power series of s.

4 NON EXISTENCE OF COMPLETELY PERIODIC SOLUTIONS FOR FOS

Theorem 2 Suppose that $f(t)$ is a non-constant periodic function with period T , i.e. $f(t) = f(t+T)$, for all $t > 0$. If $f(t)$ is $n-$times differentiable, function ${}^{C}D_t^{\alpha}f(t)$, where $0 < \alpha < n$ and n is the first integer greater than α, cannot be a periodic function with period T .

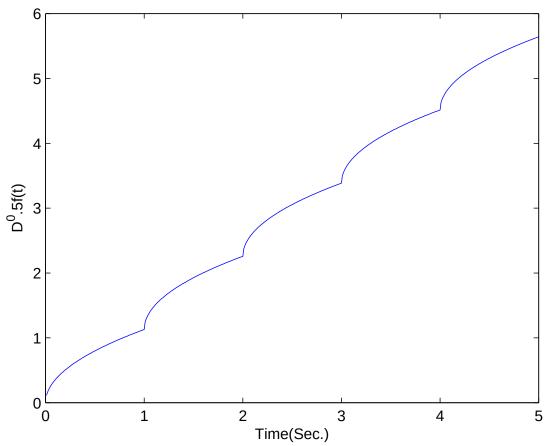

FIGURE 1. States figure of $D^{\frac{1}{2}}f(t), f(t) = t, 0 \leq t < 1, f(t+1) = f(t)$

Proof

Taking Laplace transform for ${}^{C}D_t^{\alpha}f(t)$, from Property 1,we have

$$\mathbf{L}({}_0^C D_t^{\alpha} f(t)) = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^k(0),$$

As $F(s)$ can be expressed as a series of s, then ${}^{C}D_t^{\alpha}f(t)$ can be be expressed as a series of s unless α is an integer number. By the way, any definition of fractional derivative hold the above conclusion.

For simplicity, suppose $D_t^{\alpha}f(t) = {}^{C}D_t^{\alpha}f(t)$. From Fig. 2, we can see the curve of $D^{\frac{1}{2}}f(t)$,
$f(t) = t, 0 \leq t < 1, f(t+1) = f(t)$ is aperiodic but monopoly increasing.

Example 2 By Property 3, it follows that

$$D^{\alpha}\sin(t) = t^{1-\alpha}E_{2,2-\alpha}(-t^2),$$

$$D^{\alpha}\cos(t) = t^{-\alpha}E_{2,1-\alpha}(-t^2) - \frac{t^{\alpha-1}}{\Gamma(\alpha)},$$

$$\mathbf{L}(D^{\alpha}\sin(t)) = \frac{s^{\alpha}}{s^2+1} - s^{\alpha-1}\sin(0),$$

$$\mathbf{L}(D^{\alpha}\cos(t)) = \frac{s^{\alpha+1}}{s^2+1} - s^{\alpha-1}\cos(0).$$

It is noted that $\mathbf{L}(D^{\alpha}\sin(t))$ and $\mathbf{L}(D^{\alpha}\cos(t))$ cannot be expressed as the series of s so that they are not periodic functions. From Fig 3. we can see the curves of $D^{\alpha}\sin(t), 0 < \alpha < 1$ are aperiodic.

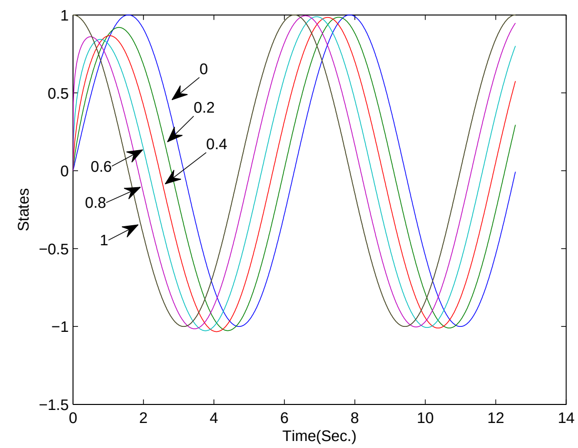

FIGURE 2. States figure of $D^{\alpha}\sin(t)$, for parameter α varying from 0.2 to 1

Theorem 3 For fractional order periodic system

$$_0^C D_t^{\alpha} x(t) = A(t)x(t), \quad (1)$$

where $A(t+T) = A(t)$, then this system has no completely periodic solution which means that the each sub vector $x_i(t)$ is periodic.

Proof By contradiction. Suppose the system has a completely periodic solution $x(t)$, where $x(t+T) = x(t)$. Then, we have that $y(t) = A(t)x(t)$ is a periodic vector function and the Laplace transform of $y(t)$ is denoted as $Y(s)$.

Taking the Laplace transform for periodic system

$$_0^C D_t^{\alpha} x(t) = A(t)x(t),$$

from Property 1 we get

$$s^{\alpha}X(s) - s^{\alpha-1}x(0) = Y(s).$$

The left of the above equation is a fractional power series of s, whereas the right of the above is an integer power series of s. This means for non zero periodic $x(t)$ can not be a solution of fractional order periodic system.

The system equation in Theorem 7 can be extended to time varying nonlinear systems.

Theorem 4 For a fractional order nonlinear system

$$ {}_0^C D_t^{\alpha} x(t) = f(x(t)), $$

then this system has no completely periodic solution.

Proof The proof is similar to the proof of Theorem 7 and is omitted.

Example 3 Linear time invariant fractional system with order $0 < \alpha < 1$, in Theorem 6 with parameter $A(t) = A$ has no completely periodic solution.

Proof By contradiction. Suppose the linear time invariant fractional system has a periodic solution. From Theorem 7, taking Laplace transform for

$$ {}^C D^{\alpha} x(t) = Ax(t), $$

we have that

$$ s^{\alpha} X(s) - s^{\alpha-1} x(0) = AX(s). $$

If we denote

$$ F(s) = s^{\alpha} X(s) - s^{\alpha-1} x(0), $$

then $F(s)$ cannot be expressed as the integer power series of s but $AX(s)$ can. And $f(t)$ is aperiodic function but $Ax(t)$ is a periodic function. So, the system can not have any completely periodic solution.

5 EXISTENCE OF PARTLY PERIODIC SOLUTIONS OF FOS

For integer order system, it is easy to give examples to show that not only there exist completely periodic solutions but also there exist partly periodic solutions.

Example 4 An example owning completely periodic solutions of integer order periodic for system

$$ \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t). \quad (2) $$

Example 5 An example owning partly periodic solutions of integer order periodic for system

$$ \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} x(t). \quad (3) $$

From Section 4, we have that there do not exist completely periodic solutions for fractional order periodic system (1). For system (1) it follows from Property 3 that if all sub-vectors $x(t)$ in system (1) are periodic then ${}^C D^{\alpha} x(t)$ can not be periodic, vice versa. Wether there exist partly periodic solutions of fractional order periodic system (1) or not is still an open problem. In the following discussion, we construct an example according to Property 3 to show that there do exist partly periodic solutions of fractional order periodic system (1). From Property 3 we know that if sub-vectors $x_i(t)$ in system (1) are periodic then ${}^C D^{\alpha} x_i(t)$ can not be periodic, vice versa. Based on Property 3, we can construct the following valid example of owning partly periodic solutions by simply setting $x_{i+1} = {}^C D^{\alpha} x_i(t)$.

Example 6 An example owning partly periodic solutions of fractional order periodic for system

$$ {}^C D^{0.5} x(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} x(t). \quad (4) $$

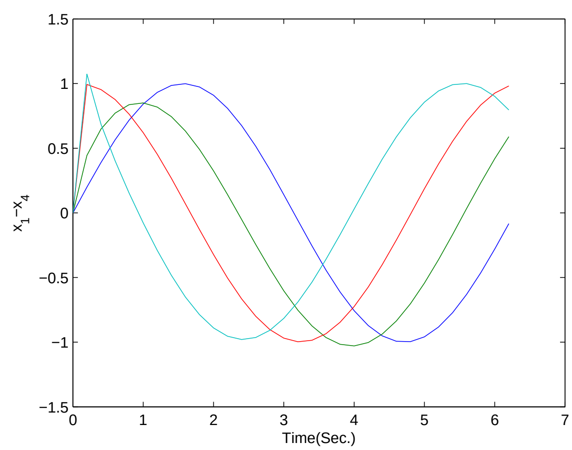

FIGURE 3. States figure of $x_1 - x_4$ for Example 6

From Fig. 3, we can see that the states $x_1 = \sin(t)$ and $x_3 = \cos(t)$ are periodic but $x_2 = D^{0.5}\sin(t)$ and $x_2 = D^{0.5}\cos(t)$ are aperiodic, respectively.

6 CONCLUSIONS

In this paper, it is proved that the fractional-order derivatives (obtained based on the Grunwald-Letnikov definition, Riemann-Liouville definition, or Caputo definition) of a periodic function with a specific period cannot be a periodic function with the same period. Based on this proved statement, it is concluded that the existence of completely periodic solutions in autonomous fractional order systems is impossible. By using the proof given by the property of Laplace transform a remarkable property for fractional-order derivatives of periodic functions is presented in this paper. The time varying fractional order systems do not have any completely periodic solution. A basic and simple proof approach of nonexistence of the periodic solution for dynamic fractional order systems is given by a new property of Laplace transform. The examples and state figures are given to illustrate the method presented. The method presented in paper deduces not only the non existence of completely periodic solutions of periodic fractional order systems but also helps to find the partly periodic solutions. The existence of partly periodic solutions for dynamic fractional order systems is the future research topic.

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