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Regional detection of unknown sources for the sub-diffusion process

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Abstract— **In this paper, we shall discuss the problem of regional detection of unknown source in a sub-diffusion process. The notions of sources, detection and regional strategic/spy sensors are introduced and the relationships between regional spy sensors and regional strategic sensors in a sub-diffusion process are explored. Moreover, we show, using an extension of the Hilbert Uniqueness Methods approach, how to reconstruct a regionally detectable source for the time fractional diffusion system based on the output functions. An example is given in the end to confirm our results.**

Index Terms— **Regional detection; sources; sensors; sub-diffusion process; time fractional diffusion systems**

I. INTRODUCTION

Recently the environmental problems, especially the pollution problems have drawn increasing attention due to those past natural disasters and their risks [1], [2]. Moreover, for the pollution problems, people often worry about their expansion and are aware that their danger will increase if the source is still unknown. As cited in [3] (see also Definition 1 below), a source can be characterized by three parameters according to its (pointwise, zone, boundary, fixed or moving), intensity and life duration. The source can be regarded as an unknown control to be detected by sensors. In addition, by [4], [5], we see that the time fractional diffusion systems can be used to well describe those sub-diffusion processes, which offer better performance not achievable before using conventional diffusion systems and surely raise many potential research opportunities at the same time. So in this paper, we shall study the problem of regional detection of unknown sources in the sub-diffusion system, which can be considered as an extension of the previous work (see [3], [6] and [7] for example).

It is well known that for diffusion system, in general, not all the sources can be detected in the whole domain of interest. Then here we try to introduce the notion of regional detection of unknown sources, where we are interested in the detection and reconstruction of a source only in a subregion of the whole domain. As it will be shown, the idea of regional detection can significantly save energy resources. Moreover, it is easier to detect a source in a subregion even if for those cases where we have a possibility to detect whole domain. To the best of our knowledge, no result is available on this topic. We hope that the results here could provide some insights into the qualitative analysis of the study of fractional diffusion systems.

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The rest of this paper is organized as follows. The second section is devoted to the mathematical concepts of source and detection. The third section is focused on the regional strategic sensors, regional spy sensors and their relationships. In Section 4, our main results on the reconstruction of a regionally detectable source from the output equation for the time fractional diffusion system is obtained by using an extension of the Hilbert Uniqueness Methods approach. An application is worked out in the end to illustrate our obtained results.

II. DEFINITION: PRELIMINARIES

In this section, we shall introduce the notions of sources, detection and some preliminary results, which play key role to obtain our main results.

A. Sources

Let Ω be an open bounded subset of $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ and we state the following definitions of a source.

Definition 1: [3] A source S is characterized by a triplet (Σ, g, I), where

1) $\Sigma(\cdot): t \in I \to \Sigma(t) \subseteq \Omega$ represents the support of source that maybe vary in time t;
2) $g(t,\cdot): \xi \in \Sigma(t) \to g(t,\xi)$ defines the intensity of source in ξ at time t;
3) $I = \{t : g(t,\cdot) \neq 0 \text{ on } \Sigma(t)\}$ denotes the support of g and represents the life duration of source S;

Generally, it is supposed that $I := [0,b]$. In particular, if I is a union of several intervals, we say that the studied system is excited by consecutive sources. Moreover, note that if we extend Σ and g of (Σ, g, I) as follows

$$g(t,\cdot) = \begin{cases} g(t,\cdot) & \text{on } I \\ 0 & \text{else} \end{cases} \quad \text{and } \Sigma(t) = \begin{cases} \Sigma(t) & \text{on } I \\ \emptyset & \text{else,} \end{cases} \qquad (1)$$

we get that (Σ, g, I) is well defined on whole I.

A source S is said to be pointwise (respectively, zone) if its support $\Sigma(t)$ is reduced to a single point (respectively, a region) of Ω for all t in I. This is the case of a moving pointwise (respectively, zone) source. The source is said to be fixed if Σ is not depending on time, which may be a zone or pointwise. Moreover, it is obvious that a source can be on the boundary ($\Sigma(t) \subseteq \Gamma,\ \forall t \in I$), and in this case, we can define the similar pointwise/zone fixed or moving boundary sources. Finally, we note that when discussing the detection problem, the pointwise fixed source defined as $\Sigma(t) = \{\sigma\} \in \Omega$ is usually used.

B. Detection

Let us consider the following time fractional diffusion system

$$\begin{cases} {}_0D_t^{\alpha} y(t) = Ay(t) + S(t), \quad t \in I, \ 0 < \alpha \leq 1, \\ \lim\limits_{t \to 0^+} {}_0I_t^{1-\alpha} y(t) = y_0, \end{cases} \tag{2}$$

where $S = (\Sigma, g, I)$ is a source supposed to be unknown, A is the infinitesimal generator of a strongly continuous semigroup $\{\Phi(t)\}_{t \geq 0}$ in the Hilbert space $L^2(\Omega)$ and $y_0 \in L^2(\Omega)$. In addition, $y \in L^2(0,b;V)$, where V is a Hilbert space such that

$$V^* \subseteq L^2(\Omega) \subseteq V$$

with continuous injections (V^* is the dual of V). Here ${}_0D_t^{\alpha}$ and ${}_0I_t^{\alpha}$ denote the Riemann-Liouville fractional derivative and integral respectively, given by [8], [9]

$$\begin{aligned} {}_0D_t^{\alpha} y(t) &= \frac{\partial}{\partial t} {}_0I_t^{1-\alpha} y(t), \ 0 < \alpha \leq 1 \quad \text{and} \\ {}_0I_t^{\alpha} y(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \ \alpha > 0. \end{aligned}$$

The output function is given as follows

$$z(t) = Cy(t), \tag{3}$$

where $C \in \mathscr{L}\left(L^2(0,b;V), L^2(0,b;Z)\right)$ and Z is a Hilbert space.

Definition 2: A source S is said to be detectable provided that it can be reconstructed from the system (2) and the output function (3).

Note that the detection of the source can be done by neglecting its life duration, here only consider the source as a couple (Σ, g). Denoted $\mathscr{E}$ the set of such sources, one has

$$\mathscr{E} \subseteq \mathscr{F}\left(0,b;\mathscr{P}(\Omega)\right) \times \mathscr{F}\left(0,b;V\right), \tag{4}$$

where $\mathscr{P}(\Omega)$ is the set of parts of Ω, $\mathscr{F}(0,b;*)$ is the space of functions $f : [0,b] \to *$ and here $\mathscr{E}$ may be considered as a vector space with convenient scalar product operations. By Definition 2, it is not difficult to see the following remark.

Remark 1: A source S is said to be detectable on I if the knowledge of system (2), together with the output function (3) is sufficient to guarantee that the operator

$$Q : S \in \mathscr{E} \to z \in L^2(0,b;Z) \tag{5}$$

is injective, where z is the observation corresponding to source S.

However, in practice, the reconstruction of all parameters of a source seems to be difficult (or impossible). Then here we only detect some parameters of source. This is to say that the source is regionally detectable.

Let ω be a non-empty, not necessarily connected subregion of Ω. Consider the subspace

$$\mathscr{E}_{\omega} = \{(\Sigma, g) \in \mathscr{E} : \Sigma(t) \subseteq \omega, \ \forall t \in I\} \tag{6}$$

and the operator

$$Q_{\omega} : S \in \mathscr{E}_{\omega} \to z \in L^2(0,b;Z), \tag{7}$$

now we are ready to state the following definition.

Definition 3: A source S is called to be $\omega-$detectable on $[0,b]$ if the knowledge of system (2), together with output function (3) is sufficient to guarantee that Q_{ω} is injective.

Remark 2: If

$$\omega_1 \subseteq \omega_2 \subseteq \Omega \text{ with } \Sigma(t) \subseteq \omega_1, \tag{8}$$

then we get that the source is ω_1-detectable provided that it is ω_2-detectable

III. Regional strategic sensors and regional spy sensors

The aim of this section is to explore the notion of regional strategic sensors, regional spy sensors and the relationships.

A. Regional strategic sensors

In this part, let us consider the following autonomous system

$$\left.\begin{aligned} &{}_0D_t^{\alpha} y(t) = Ay(t), \quad t \in I, \\ &\lim_{t \to 0^+} {}_0I_t^{1-\alpha} y(t) = y_0 \text{ supposed to be unknown.} \end{aligned}\right\} \tag{9}$$

From [10], [11], it follows that

$$y(t) = t^{\alpha-1} K_{\alpha}(t) y_0(x), \tag{10}$$

where

$$K_{\alpha}(t) = \alpha \int_0^{\infty} \theta \phi_{\alpha}(\theta) \Phi(t^{\alpha}\theta) d\theta, \tag{11}$$

$$\phi_{\alpha}(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \psi_{\alpha}(\theta^{-\frac{1}{\alpha}}) \text{ and} \tag{12}$$

$$\psi_{\alpha} := \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-\alpha n - 1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha) \tag{13}$$

is a probability density function such that [10], [12]

$$\int_0^{\infty} \psi_{\alpha}(\theta) d\theta = 1 \text{ and } \int_0^{\infty} \theta^{\nu} \phi_{\alpha}(\theta) d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}, \tag{14}$$

$\nu \geq 0$. Then the output function becomes

$$z(t) = Cy(t) = K(t) y_0, \tag{15}$$

where $K(t) := Ct^{\alpha-1} K_{\alpha}(t)$.

Let $p_{\omega} : L^2(\Omega) \to L^2(\omega)$ defined by $p_{\omega} y = y|_{\omega}$ is the projection operator in ω and

$$p_{\omega}^* y(x) := \begin{cases} y(x), & x \in \omega, \\ 0, & x \in \Omega \backslash \omega \end{cases} \tag{16}$$

is its adjoint operator. We now give the following definition.

Definition 4: [13], [14] The system (9) with (15) is said to be $\omega-$weakly observable if

$$Ker(Kp_{\omega}^{*}) = \{0\}. \tag{17}$$

Moreover, to state our main results, it is supposed that the measurements are made by p sensors $(D_i, f_i)_{1\le i\le p}$ and then the out function becomes

$$z(t) = \Big((f_1, y(t))_{L^2(D_1)}, \cdots, (f_p, y(t))_{L^2(D_p)}\Big)^T, \tag{18}$$

where $t \in I$ and $(\cdot,\cdot)_{L^2(\Omega)}$ is the inner product of space $L^2(\Omega)$. We then see the following definition.

Definition 5: Sensors $(D_i, f_i)_{1\le i\le p}$ are said to be $\omega-$strategic if the system (9) + (18) is $\omega-$weakly observable.

Moreover, for operator A, assume that we can find a sequence $(\lambda_j, \xi_{jk}) : k = 1,2,\cdots,r_j,\ j = 1,2,\cdots$ such that

- For each $j = 1,2,\cdots$, λ_j is the eigenvalue of A with multiplicities r_j and
$$0 > \lambda_1 > \lambda_2 > \cdots > \lambda_j > \cdots, \quad \lim_{j\to\infty} \lambda_j = -\infty.$$
- For each $j = 1,2,\cdots$, ξ_{jk} $(k = 1,2,\cdots,r_j)$ is the orthonormal eigenfunction corresponding to λ_j, i.e.,
$$(\xi_{jk_m}, \xi_{jk_n})_{L^2(\Omega)} = \begin{cases} 1, & k_m = k_n, \\ 0, & k_m \neq k_n, \end{cases}$$
where $1 \le k_m, k_n \le r_j$, $k_m, k_n \in \mathbf{N}$.

We are now ready to state the following result.

Theorem 1: Define $p \times r_j$ matrices G_j as

$$G_j = \begin{bmatrix} \xi_{j1}^1 & \xi_{j2}^1 & \cdots & \xi_{jr_j}^1 \\ \xi_{j1}^2 & \xi_{j2}^2 & \cdots & \xi_{jr_j}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_{j1}^p & \xi_{j2}^p & \cdots & \xi_{jr_j}^p \end{bmatrix}_{p\times r_j}, \tag{19}$$

where $\xi_{jk}^i = (\xi_{jk}, f_i)_{L^2(D_i)}$, $i = 1,2,\cdots,p$ and $k = 1,2,\cdots,r_j$. Then the sensors $(D_i, f_i)_{1\le i\le p}$ are $\omega-$strategic if and only if

$$p \ge r = \max\{r_j\} \quad \text{and} \tag{20}$$

$$rank\ G_j = r_j \text{ for all } j = 1,2,\cdots. \tag{21}$$

Proof. It follows from Definition 4 that the sensors $(D_i, f_i)_{1\le i\le p}$ are $\omega-$strategic if and only if

$$Ct^{\alpha-1}K_\alpha(t)p_\omega^* y = 0 \Rightarrow y = 0, \quad \forall y \in L^2(\omega).$$

Moreover, by [11], we see that

$$\alpha E_{\alpha,\beta}^2 = E_{\alpha,\beta-1} - (1+\alpha-\beta)E_{\alpha,\beta}, \tag{22}$$

where

$$E_{\alpha,\beta}^{\mu}(z) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, z \in \mathbf{C}, \alpha, \beta, \mu \in \mathbf{C}, \mathbf{Re}\ \alpha > 0$$

is known as the generalized Mittag-Leffler function in three parameters. In particular, when $\mu = 0$, write $E_{\alpha,\beta}^0(z) = E_{\alpha,\beta}(z)$ for short and when $\mu = 0$, $\beta = 1$, write $E_{\alpha,1}(z) = E_\alpha(z)$ for short.

Then we have

$$\begin{aligned} & K_\alpha(t)p_\omega^* y \\ = & \ \alpha \int_0^\infty \theta \phi_\alpha(\theta) \Phi(t^\alpha \theta) p_\omega^* y d\theta \\ = & \ \alpha \int_0^\infty \theta \phi_\alpha(\theta) \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} \exp(-\lambda_j t^\alpha \theta)(p_\omega^* y, \xi_{jk})\xi_{jk}(x) d\theta \\ = & \ \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} \alpha E_{\alpha,\alpha+1}^2(-\lambda_j t^\alpha)(p_\omega^* y, \xi_{jk})\xi_{jk}(x) \\ = & \ \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} E_{\alpha,\alpha}(\lambda_j t^\alpha)(p_\omega^* y, \xi_{jk})_{L^2(\Omega)}\xi_{jk}. \end{aligned}$$

Consequently, the necessary and sufficient condition for strategic sensors $(D_i, f_i)_{1\le i\le p}$ is that

$$\sum_{j=1}^{\infty} \sum_{k=1}^{r_j} t^{\alpha-1} E_{\alpha,\alpha}(\lambda_j t^\alpha) \xi_{jk}^i (p_\omega^* y, \xi_{jk})_{L^2(\Omega)} = 0,$$

for all $i = 1,2,\cdots,p$ can imply

$$y = 0 \text{ on } L^2(\Omega), \tag{23}$$

i.e.,

$$\sum_{j=1}^{\infty} \frac{E_{\alpha,\alpha}(\lambda_j t^\alpha)}{t^{1-\alpha}} G_j (p_\omega^* y, \xi_{jk})_{L^2(\Omega)} = \theta := (0,\cdots,0) \in \mathbf{R}^p \tag{24}$$

can imply

$$y = 0. \tag{25}$$

Finally, since $E_{\alpha,\alpha}(\lambda_j t^\alpha) > 0$ for all $t \ge 0$, $j = 1,2,\cdots$, we then show our proof by using the Reductio and Absurdum.

(a) Necessity. If $p \ge r = \max\{r_j\}$ and $rank\ G_j < r_j$ for some $j = 1,2,\cdots$, then we can find a element $\tilde{y} \in L^2(\Omega)$, $\tilde{y} \neq 0$ satisfying

$$G_j(p_\omega^* \tilde{y}, \xi_{jk})_{L^2(\Omega)} = 0. \tag{26}$$

Thus, there exists a nonzero element $\tilde{y} \in L^2(\Omega)$ satisfying

$$\sum_{j=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(\lambda_j t^\alpha) G_j (p_\omega^* \tilde{y}, \xi_{jk})_{L^2(\Omega)} = \theta, \tag{27}$$

which implies that the sensors $(D_i, f_i)_{1\le i\le p}$ are not $\omega-$strategic.

(b) Sufficiency. If the sensors $(D_i, f_i)_{1\le i\le p}$ are not strategic, by Definition 5, we can find a element $\hat{y} \neq 0$, $\hat{y} \in L^2(\Omega)$ such that

$$G_{j^*}(p_\omega^* \hat{y}, \xi_{j^*k})_{L^2(\Omega)} = 0 \tag{28}$$

for some $j^* = 1,2,\cdots$. Consequently, if $p \ge r = \max\{r_j\}$, it is sufficient to see that $rank\ G_{j^*} < r_j$. The proof is complete.

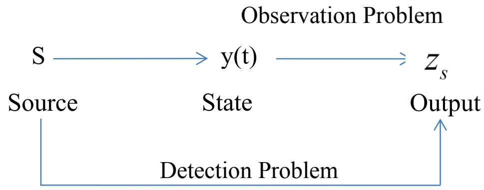

Fig. 1. The relationship between detection problem and observation problem

B. Regional spy sensors

Consider the system (2) and suppose that the measurements are given by p sensors $(D_i, f_i)_{1\leq i\leq p}$, in this case, the out function $z(t)$ corresponding S becomes

$$z(t) = (z_1(t), z_2(t), \cdots, z_q(t))^T \in \mathbf{R}^p, \quad t \in I, \tag{29}$$

where

$$\begin{aligned} z_i(t) = & \sum_{j=1}^{\infty}\sum_{k=1}^{r_j} \frac{E_{\alpha,\alpha}(\lambda_j t^\alpha)}{t^{1-\alpha}} (\xi_{jk}, y_0)_{L^2(\Omega)} \xi_{jk}^i \\ & + \sum_{j=1}^{\infty}\sum_{k=1}^{r_j} \int_0^t \frac{E_{\alpha,\alpha}(\lambda_j (t-\tau)^\alpha)}{(t-\tau)^{1-\alpha}} (S(\tau), \xi_{jk})_{L^2(\Omega)} d\tau \xi_{jk}^i \end{aligned} \tag{30}$$

and $\xi_{jk}^i = (\xi_{jk}, f_i)_{L^2(D_i)}$. Now we state the following definition on regional spy sensors, which may lead to numerous problems and pose challenging research topics.

Definition 6: Sensors are said to be $\omega-$spy sensors if they can detect any unknown sources in $\mathscr{E}_\omega \subseteq \mathscr{E}$.

C. The relationships between $\omega-$spy sensors and $\omega-$strategic sensors

Fig.1 shows that the detection problem and observation problem are different, which leads immediately to the difference between $\omega-$strategic sensors and $\omega-$spy sensors.

Lemma 1: Strategic ($\omega-$strategic) sensors are spy ($\omega-$spy) sensors, while the converse is not true.
Proof. Based on the conclusion that $S \to y(t)$ is injective but not surjective in [13] (see also in [14]), it is not difficult to see that if sensors are $\omega-$strategic, they are $\omega-$spy sensors, while the converse fails. Here ω may be whole domain. The proof is finished.

Moreover, considering that (2) is a line system, based on the Proposition 3.1 in [15], it suffices to assume that $y_0 = 0$ in the following discussion. In particular, if we, for example, consider a zone persistent sorce $S = (\Sigma, g, I) \in \mathscr{E}_\omega$. When $g \in S$ and $g \in L^2\left(0, +\infty; L^2(\Omega)\right)$, we can obtain the following result.

Theorem 2: Suppose that $g \in S$ and $g \in L^2\left(0, +\infty; L^2(\Omega)\right)$. Then $(D_i, f_i)_{1\leq i\leq p}$ are $\omega-$spy sensors if and only if they are $\omega-$strategic sensors.
Proof. If the sensors $(D_i, f_i)_{1\leq i\leq p}$ are $\omega-$strategic, it then follows from Lemma 1 that they are $\omega-$spy sensors.

Conversely, if the sensors $(D_i, f_i)_{1\leq i\leq p}$ are not $\omega-$strategic, by Theorem 1, we can find an element $\hat{y} \neq 0$, $\hat{y} \in L^2(\omega)$ such that for some $j^* = 1, 2, \cdots,$

$\sum_{k=1}^{r_j} \xi_{j^*k}^i (p_\omega^* \hat{y}, \xi_{j^*k})_{L^2(\Omega)} = 0$ holds for all $i = 1, 2, \cdots, p$.

Moreover, by Definition 6, the necessary and sufficient condition to the $\omega-$spy sensors $(D_i, f_i)_{1\leq i\leq p}$ is that for any unknown sources $S \in \mathscr{E}_\omega \subseteq \mathscr{E}$, the operator

$$Q_\omega : \begin{array}{l} \mathscr{E}_\omega \to L^2(0, b; Z) \\ S \to z(t) = Cy(t) = (z_1(t), z_2(t), \cdots, z_q(t))^T \in \mathbf{R}^p \end{array} \tag{31}$$

is injective, where

$$z_i(t) = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j} \int_0^t \frac{E_{\alpha,\alpha}(\lambda_j (t-\tau)^\alpha)}{(t-\tau)^{1-\alpha}} \xi_{jk}^i (S(\tau), \xi_{jk})_{L^2(\Omega)} d\tau,$$

$i = 1, 2, \cdots, p$. Then we see

$$Q_\omega p_\omega^* \hat{y} = 0 \text{ and } \hat{y} \not\equiv 0. \tag{32}$$

Therefore, since $S \in \mathscr{E}_\omega$ and $g \in L^2\left(0, +\infty; L^2(\Omega)\right)$, let

$$\widehat{g} = g + p_\omega^* \hat{y},$$

one has

$$Q_\omega \widehat{S} = Q_\omega S, \tag{33}$$

where $\widehat{S}$ is the source having $\widehat{g}$ as its intensity. This means that $(D_i, f_i)_{1\leq i\leq p}$ are not $\omega-$spy sensors and the proof is finished.

Remark 3: The obtained results can be extended to the cases of pointwise or boundary sensors.

IV. Reconstruction of a Regionally Detectable Source

In this section, we show how to reconstruct a source $S \in \mathscr{E}_\omega$ under the hypothesis that the system studied is regionally detectable.

Consider again system (2) and the output (3), suppose that operator Q_ω is injective. Then the semi-norm defined by

$$\|S\|_{F_\omega} = \|Q_\omega S\|_{L^2(0,b;L^2(\Omega))}, \quad S \in \mathscr{E}_\omega \tag{34}$$

is a norm. So $F_\omega := \overline{\mathscr{E}_\omega}$ is a Hilbert space with the inner product

$$\langle S_1, S_2 \rangle_{F_\omega} = \langle Q_\omega S_1, Q_\omega S_2 \rangle_{L^2(0,b;L^2(\Omega))}. \tag{35}$$

Consider the operator $\Lambda_\omega : F_\omega \to F_\omega^*$ as follows

$$\begin{aligned} & \Lambda_\omega S = Q_\omega^* Q_\omega S \\ & = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j} \int_\cdot^T \left[\begin{array}{l} \frac{E_{\alpha,\alpha}(\lambda_j^*(r-\cdot)^\alpha)}{(r-\cdot)^{1-\alpha}} C^* C \times \\ \int_0^r \frac{E_{\alpha,\alpha}(\lambda_j (r-\tau)^\alpha)}{(r-\tau)^{1-\alpha}} S_{jk}(\tau) d\tau \end{array} \right] dr \xi_{jk}^i, \end{aligned} \tag{36}$$

where $S_{jk}(\tau) = (S(\tau), \xi_{jk})_{L^2(\Omega)}$, Q_ω^* is the adjoint operator of Q_ω. One then has the following result.

Lemma 2: Λ_ω has a unique extension as an isomorphism from F_ω into its dual F_ω^*.

Proof. It follows from Eq. (41) and (36) that

$$\langle \Lambda_\omega S_1, S_2 \rangle_{L^2(0,b;L^2(\Omega))} = \langle S_1, S_2 \rangle_{F_\omega}. \tag{37}$$

Then if we consider the linear mapping

$$\Lambda_\omega^{S_1}: \begin{array}{l} \overline{\mathscr{E}_\omega} \to \mathbf{R} \\ S_2 \to \langle \Lambda_\omega S_1, S_2 \rangle_{L^2(0,b;L^2(\Omega))}, \end{array} \tag{38}$$

we get that

$$\left| \left\langle \Lambda_\omega^{S_1} S_1, S_2 \right\rangle_{L^2(0,b;L^2(\Omega))} \right| \le \|S_1\|_{F_\omega} \|S_2\|_{F_\omega}.$$

Therefore, $\Lambda_\omega^{S_1}$ is a continuous operator and has a unique extension to F_ω. Then $\Lambda_\omega^{S_1} S_1 \in F_\omega^*$ and

$$\|\Lambda_\omega^{S_1}\|_{F_\omega^*} = \|S_1\|_{F_\omega}, \quad \forall S_1 \in \mathscr{E}_\omega. \tag{39}$$

Moreover, by (39), the linear operator $\Lambda_\omega : F_\omega \to F_\omega^*$ is a continuous operator from F_ω and then can be extended to F_ω. Hence, from Eq. (37) and (39), we get that $\Lambda_\omega : F_\omega \to F_\omega^*$ is an isomorphism. The proof is finished.

Now we are ready to state the following theorem.

Theorem 3: If Q_ω is injective, the source S is obtained from the corresponding observation z as the unique solution of the equation

$$\Lambda_\omega S = Q_\omega^* z. \tag{40}$$

Proof. From the argument above, we get that if Q_ω is injective, then $\|S\|_{F_\omega}$ is a norm and $F_\omega = \overline{\mathscr{E}_\omega}$ is a Hilbert space with the inner product

$$\langle S_1, S_2 \rangle_{F_\omega} = \langle Q_\omega S_1, Q_\omega S_2 \rangle_{L^2(0,b;L^2(\Omega))}. \tag{41}$$

Based on the Theorem 1.1 in [16], to complete our proof, we only need to show that Λ_ω is coercive operator, i.e., there exists a positive constant γ such that

$$\langle \Lambda_\omega S, S \rangle_{F_\omega} \ge \gamma \|S\|_{F_\omega}^2, \quad \forall S \in F_\omega. \tag{42}$$

In fact, for any $S \in F_\omega$, we have

$$\begin{aligned} \langle \Lambda_\omega S, S \rangle_{F_\omega} &= \langle Q_\omega^* Q_\omega S, S \rangle_{F_\omega} \\ &= \langle Q_\omega S, Q_\omega S \rangle_{L^2(0,b;L^2(\Omega))} \\ &= \|S\|_{F_\omega}. \end{aligned} \tag{43}$$

Then (40) has a unique solution, which is also the unique solution of the source. The proof is complete.

V. An example

Let $\Omega = [0,1]$, $\omega \subseteq \Omega$ and consider the following system

$$\begin{cases} {}_0D_t^\alpha y(x,t) = \frac{\partial^2}{\partial x^2} y(x,t) + g(t,\eta) \quad \text{in} \quad \Omega \times [0,b], \\ y(0,t) = y(1,t) = 0 \quad \text{on} \quad [0,b], \\ \lim\limits_{t \to 0^+} {}_0I_t^{1-\alpha} y(x,t) = 0 \quad \text{in} \quad \Omega, \end{cases} \tag{44}$$

where g is the extensity of the source $S = (\Sigma, g, I)$. For Laplace operator $\triangle = \frac{\partial^2}{\partial x^2} y(x,t)$, $\xi_i(x) = \sqrt{2}\sin(i\pi x)$ is the orthonormal basis of $\triangle$ corresponding the eigenvalue $\lambda_i = -i^2\pi^2$. Then we have

$$\begin{aligned} &z(t) = Cy(t) \\ &= C \sum_{i=1}^{\infty} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda_i (t-\tau)^\alpha) (g(\tau,\eta), \xi_i)_{L^2(\Omega)} d\tau \xi_i. \end{aligned}$$

Assume that the source S is a zone sensor and independent of time, then we have $g(t,\eta) \equiv g(\eta)$, $\forall \eta \in \omega$. Moreover, suppose that the system (44) is observed by an $\omega-$ spy sensor (D, f) in Ω. In this case, we get that the operator Q_ω is injective and given by

$$\begin{aligned} &(Q_\omega S)(t) \\ &= \sum_{i=1}^{\infty} \int_0^t \frac{E_{\alpha,\alpha}(\lambda_i (t-\tau)^\alpha)}{(t-\tau)^{1-\alpha}} (g(t), \xi_i)_{L^2(\omega)} d\tau \ (\xi_i, f)_{L^2(D)}. \end{aligned} \tag{45}$$

By Theorem 2, we see that if g in Eq. (45) satisfying $g \in L^2\left(0, +\infty; L^2(\Omega)\right)$, then (D, f) is $\omega-$strategic sensor.

Moreover, since the adjoint operator of Q_ω is

$$\begin{aligned} &(Q_\omega^* z)(t) \\ &= \sum_{j=1}^{\infty} \int_t^T \frac{E_{\alpha,\alpha}(\lambda_j (r-t)^\alpha)}{(r-t)^{1-\alpha}} z(r) dr \ (\xi_j, f)_{L^2(D)} p_\omega \xi_j. \end{aligned} \tag{46}$$

We get that

$$\begin{aligned} &(\Lambda_\omega S)(t) \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \left(\int_t^T \int_0^r \frac{E_{\alpha,\alpha}(\lambda_j (r-t)^\alpha)}{(r-t)^{1-\alpha}} \frac{E_{\alpha,\alpha}(\lambda_i (r-\tau)^\alpha)}{(r-\tau)^{1-\alpha}} (g(r), \xi_i)_{L^2(\omega)} d\tau dr \right) \\ &\qquad \times (\xi_i, f)_{L^2(D)} (\xi_j, f)_{L^2(D)} p_\omega \xi_j. \end{aligned}$$

Then by Lemma 2, Λ_ω has a unique extension as an isomorphism from $F_\omega := \overline{\mathscr{E}_\omega}$ into its dual F_ω^*. Moreover, it follows from Theorem 3 that the source S can be obtained from the corresponding observation z as the unique solution of the equation $\Lambda_\omega S = Q_\omega^* z$.

VI. Conclusions

The aim of this paper is to discuss an extension of the results in [3], [7] on the detection of distributed parameter systems. Here we mainly investigate the problem of regional detection for sub-diffusion process, which is used to characterize those unknown source. The characteristic of regional strategic/spy sensors, their relationships and the reconstruction of a regionally detectable source are explored. This situation occurs in many practical systems where we may be concerned with the knowledge of the state only in a critical subregion.

Moreover, the results presented here can be extended to more complex fractional order distributed parameter systems. For instance, the problem of spy sensors configurations of time-space fractional diffusion systems are of great interest. For more information on those potential topics, please refer to [17].

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