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SPREADING CONTROL OF SUB-DIFFUSION PROCESSES

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***Abstract* — Inspired by many real-world applications, this paper is concerned with the concept of spreadability for a class of sub-diffusion processes. The spreading control problems of sub-diffusion processes are formulated. An approach to guarantee the existence of solution to the adaptive spreading control problem based on the regional control results is proposed. We also introduce a new equivalence of the solution for the time fractional diffusion system to support the proof of our obtained results.**

1 Introduction

In the past several decades, especially after the first introduction of the continuous time random walk (CTRW) by Montroll and Weiss in [1], a growing number of contributions have been given to the anomalous diffusion process, in which the mean squared displacement (MSD) is smaller (in the case of sub-diffusion) or bigger (in the case of super-diffusion) than that in a Gaussian diffusion process [2, 3, 4]. These anomalous diffusion processes usually occur in the spatially inhomogeneous environment. For instance, the flow through porous media microscopic processes [5], or swarm of robots moving through dense forest [6] etc. Besides, it is confirmed that the MSD of anomalous diffusion process is described by a power law of fractional exponent [7, 8] and the solution of fractional differential equations can be expressed by using the Mittag-Leffler function. Based on the properties of Mittag-Leffler function [9, 10, 11, 12], then the fractional diffusion systems may provide a natural description of non-local transport and be used to well characterize those anomalous processes [13, 14].

The aim of this paper is to explore the spreadability of sub-diffusion process, in which the subdomains of the states to the system studied obeying a spatial property are nondecreasing. Since it is the first time for us to investigate the spreadability of sub-diffusion process, we focused on analytic results in this paper first. Simulation results will be presented in a more application-oriented journal with a more realistic applications scenario. Moreover, as cited in [15, 16], the applications of spreadability to the anomalous process are rich in the environmental processes. For example, the vegetation dynamics, pollution or medical processes in the spatially inhomogeneous environment. For more information on spreadability, we refer the readers to [17, 18, 19] and the references cited therein.

Motivated by the above discussions, in this paper, using optimal control techniques and the concepts of regional analysis [20, 21], an approach leading to the solution of the adaptive spreading control problem is proposed. To the best of our knowledge, no result is available on this topic. We hope that the results here could provide some insights into the control theory of this field and be used in real-life applications.

The remainder contents of this paper are structured as follows. The spreading control theory and some preliminary results are introduced in the next section and in Section 3, the adaptive spreading control problems are investigated. At last, a conclusion is presented.

2 Spreading control theory

In this section, we state some preliminary results to be used thereafter and formulate the spreading control problems for the sub-diffusion process.

2.1 Preliminary results

Suppose that $\Omega \subseteq \mathbf{R}^n$ is an open bounded subset with certain boundary $\partial\Omega$, $I = [0,T]$ is a time interval and $L^p(0,T;Z)$ $(p \geq 1)$ is the space of Z-value Bochner integrable functions on $[0,T]$ with the norm $\|z\|_{L^p(0,T,Z)} = (\int_0^T \|z(s)\|^p ds)^{1/p}$.

Let us consider the following system

$$\begin{cases} {}_0D_t^\alpha z(t) = Az(t), \ t \in I, \ 0 < \alpha < 1, \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} z(t) = z_0 \in L^2(\Omega), \end{cases} \tag{1}$$

where A is a linear operator which is densely defined on its domain $\mathscr{D}(A) \subseteq Z := L^2(\Omega)$ and generates a strongly continuous semigroup $\{\Phi(t)\}_{t\geq 0}$, $z \in L^2(0,T;Z)$ and $z(\cdot) \in Z$. Moreover, here ${}_0D_t^\alpha$ and ${}_0I_t^\alpha$ denote the Riemann-Liouville fractional order derivative and integral, respectively, given by [22, 23]

$$ {}_0D_t^\alpha z(t) = \frac{d}{dt} {}_0I_t^{1-\alpha} z(t), \ \ 0 < \alpha < 1 \ \text{ and}$$

$$ {}_0I_t^{\alpha} z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, \ \ \alpha > 0. $$

Denotes the solution of system (1) by $z(x,t)$, where $x \in \Omega$ and $t \in I$ are respectively, the space and time variables. Let $\mathscr{P}$ be a given property which may describe a spatial constraint on the state of the system (1) in space Z and let

$$ \omega_t = \{x \in \Omega : \mathscr{P}z(x,t)\}, \ \ t \in I \tag{2} $$

be the zones where the state obeys the property $\mathscr{P}$. At the initial time $t = 0$, we have $\omega_0 = \{x \in \Omega : \mathscr{P}z(x,0)\}$. Then we state the following definition.

Definition 2.1 *The system (1) is said to be $\mathscr{P}$-spreadability from ω_0 in the time interval I if the family $\{\omega_t\}_{t\in I}$ is non-decreasing, i.e. $\omega_t \subseteq \omega_s$ for all $s \geq t$.*

As for the property $\mathscr{P}$, various cases may be considered. For example, let $S_1 \subseteq \Omega \times I \times \mathbf{R}$ be a set of constraints on $z(x,t)$. Then $\mathscr{P}z(x,t)$ can be equivalent to $(x,t,z(x,t)) \in S_1$, i.e., $\mathscr{P}z(x,t) \Leftrightarrow (x,t,z(x,t)) \in S_1$. Moreover, consider

$$ \mathscr{P}z(x,t) \Leftrightarrow z(x,t) = \rho(x,t) \tag{3} $$

with $\rho : \Omega \times I \to \mathbf{R}$ as a desired target trajectory to be tracked during the time interval I, in this case, we say that the system (1) is $\rho-$spreadability. In particular, if $\rho = 0$, the system (1) is said to be null-spreadability. What's more, similarly, we say that the system (1) is $\mathscr{P}$-resorbability if ω_t satisfies $\omega_t \supseteq \omega_s$ for all $s \geq t$.

Remark 2.1

(1) *The above definition does not imply that $\omega_T = \Omega$;*

(2) *In particular, if the system is spreadable up to Ω, in that case, the property $\mathscr{P}$ must be consistent with the boundary conditions on $\partial\Omega$;*

(3) *Here we mainly focus on the growing property of the subdomains ω_t and ignore their growth speed.*

Spreadability has also been explored from another point of view in [24] by considering the growth of the areas of ω_t and in [25] by trying to connect the spreadability to the viability of dynamical systems. However, at this moment, they do not actually lead to significant results. In the sequel, we shall investigate the spreading control of sub-diffusion processes based on Definition 2.1. For more information on the spreadability, we refer readers to [16, 15, 26] and the references therein.

2.2 Spreading control problem

The aim of this part is to discuss what is the spreading control and then state the spreading control problem.

Consider the system (1) excited by the control u as follows:

$$ \begin{cases} {}_0D_t^{\alpha} z(t) = Az(t) + Bu(t), \ \ t \in I, \ 0 < \alpha < 1, \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} z(t) = z_0 \in Z, \end{cases} \tag{4} $$

where $B : \mathbf{R}^p \to Z$ is a bounded linear operator and $u \in L^2(0,T;\mathbf{R}^p)$ is the control input. By [20, 27], let $\psi_\alpha(\theta) = \frac{1}{\pi}\sum\limits_{n=1}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \theta > 0$ be a probability density function and $\phi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\psi_\alpha(\theta^{-\frac{1}{\alpha}})$ satisfying [28, 29]

$$ \int_0^{\infty} \psi_\alpha(\theta) d\theta = 1, \int_0^{\infty} \theta^{\nu} \phi_\alpha(\theta) d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}, \nu \geq 0, $$

by using Laplace transform, let $K_\alpha(t) = \alpha\int_0^{\infty}\theta\phi_\alpha(\theta)\Phi(t^{\alpha}\theta)d\theta$, we recall that the solution of the system (1), denoted by $z(x,t,u)$, is therefore given by

$$ \begin{aligned} z(x,t,u) = {} & t^{\alpha-1}K_\alpha(t)z_0 + \\ & \textstyle\int_0^t (t-s)^{\alpha-1}K_\alpha(t-s)Bu(s)ds, \ \ t \in I. \end{aligned} \tag{5} $$

Given $\mathscr{P}$ as in Eq. (3), for any $t \in I$, define

$$ \omega_t^u = \{x \in \Omega : \mathscr{P}z(x,t,u)\}, \ u \in L^2(0,T;\mathbf{R}^p) \tag{6} $$

and then we state the following definition.

Definition 2.2 *The control u is said to be a $\mathscr{P}$-spreading control if the family $\{\omega_t^u\}_{t\in I}$ is non-decreasing, i.e. the excited system is $\mathscr{P}$-spreadable. In particular, if $\mathscr{P} = 0$, then we say that u is null-spreading control.*

From the above definition, to the best of our knowledge, it is not difficult to see the following control problems related to spreadability.

(a) The spreading control problem concerns the existence and the determination of spreading controls without concerning with the energy and the areas of spreadable zones.

(b) In the case where the set of spreading controls is not empty, it is possible to derive spreadability to the whole domain Ω in minimum time? What about the energy associated with these controls?

(c) Given a non-decreasing family of subregions $\{\sigma_t\}_{t\in I}$, do controls leading to expanding a property along the subregions σ_t exist?

Moreover, suppose that $\omega_0 = \{x \in \Omega : z_0 = 0\} \neq \emptyset$ and consider the zones

$$ \omega_t^u = \{x \in \Omega : z(x,t,u) = 0\}, \tag{7} $$

$t \in I$ and $u \in L^2(0,T;\mathbf{R}^p)$, then the spreading control problem may be equivalent to

$$ \left. \begin{array}{l} \text{find a control input } u \in L^2(0,T;\mathbf{R}^p) \text{ such that:} \\ (1)\ z(x,t,u) \text{ is the solution of the system (4),} \\ (2) \text{ the family } \{\omega_t^u\}_{t\in I} \text{ is non-decreasing.} \end{array} \right\} \tag{8} $$

This problem is very difficult and here we only try to find a control such that the system under consideration is weakly spreadable, the concept of which is developed in the next subsection.

2.3 Weak spreadability

As cited in [30], the above stronger concept of spreadability is harder to achieve. So in this subsection, we shall introduce a concept of weak spreadability and try to explore its characterization.

Define $G_t : L^2(0,T;\mathbf{R}^p) \to L^2(\Omega)$ as follows

$$G_t u = \int_0^t (t-s)^{\alpha-1} K_\alpha(t-s) B u(s) ds. \tag{9}$$

For any $v \in Z^*$, it follows from the duality relationship $\langle G_t u, v\rangle_{Z\times Z^*} = \langle u, G_t^* v\rangle_{L^2\times L^2}$ that

$$(G_t^* v)(t) = B^*(T-t)^{\alpha-1} K_\alpha^*(T-t)v, \tag{10}$$

where B^* and K_α^* are respectively, the adjoint operators of B and K_α. Taking into account that (4) is a linear system, by the Proposition 3.1 in [31], it suffices to suppose that $z_0 = 0$ in the following discussion. Now we are ready to state the following definitions.

Definition 2.3 *[30]*
(a) Given $\varepsilon > 0$ and $\mathscr{P} \in Z$, the system (4) is said to be weakly $\mathscr{P}$-spreadable with the tolerance ε, if there exists a family of subdomains $(\tilde{\omega}_t)$, $(\tilde{\omega}_t) \subseteq \mathscr{P}(\Omega)$ (where $\mathscr{P}(\Omega)$ hold for the set of parts Ω) such that
(1) $\omega_0 \subseteq \tilde{\omega}_0$;
(2) $\tilde{\omega}_t \subseteq \tilde{\omega}_s$ *for all* $t \leq s,\ t,s \in I$;
(3) $\overline{Im(G_t)} = L^2(\tilde{\omega}_t)$ *for any* $t \in I$;
(4) $\tilde{\omega}_T = \Omega$.
(b) In particular, if $\mathscr{P} = 0$, then the system (1) is weakly null-spreadable.

Remark 2.2 *(1) If the system (4) is weakly spreadable, the family $(\tilde{\omega}_t)$ is not unique and each of such choices will correspond to a different ε. In particular, if a system is spreadable, we say that it is weakly spreadable with the tolerance $\varepsilon = 0$.*
(2) The above definition may be relaxed by removing condition (4).

Definition 2.4 *Any family of subregions verifying the conditions $(1)-(3)$ of Definition 2.3 is called a spread.*

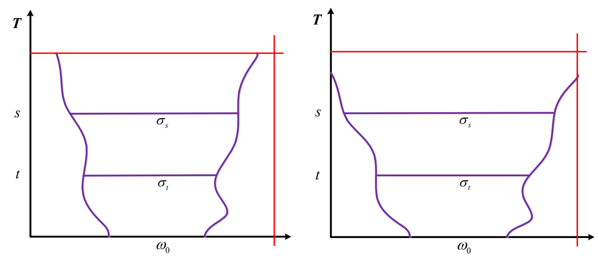
Denote S the set of spreads

$$S = \left\{ \begin{array}{r} \sigma = (\sigma_t)_{t\in I} \subseteq \mathscr{P}(\Omega) : \omega_0 \subseteq \sigma_0, \text{ and} \\ \sigma_t \subseteq \sigma_s \text{ for any } t \leq s,\ t,s \in I \end{array} \right\} \tag{11}$$

and let the set of spreads which satisfies the condition (4) of Definition 2.3 be as follows

$$S^* = \{\sigma \in S : \sigma_T = \Omega\}. \tag{12}$$

In particular, if $\Omega \subseteq \mathbf{R}^1$, see Fig. 1 for an example of spread $\sigma \in S$ or $\sigma \in S^*$.



(a) A spread in S (b) A spread in S^*

Figure 1: Spreads in S and in S^*

3 Adaptive spreading control problems

This section aims to investigate the adaptive spreading control problems by using the results of regional control theory developed in [20, 21].

3.1 Problem formulation

In this part, we first give a time-discretized formulation of the spreading control problem (8).

Let us consider a sequence $(t_i)_{0\leq i\leq m}$ of the time interval I such that $0 = t_0 < t_1 < \cdots < t_m = T$ and denote $\omega_{t_i}^{u_i}$ the associated subdomains defined by

$$\omega_{t_i}^{u_i} = \{x \in \Omega : z(x,t_i,u_i) = 0\}. \tag{13}$$

Suppose that the subregions $\omega_{t_i}^{u_i}$ are non-decreasing and given, i.e.,

$$\omega_{t_i}^{u_i} \subseteq \omega_{t_{i+1}}^{u_{i+1}}, \tag{14}$$

then the discrete version of (8) may be stated as follows:

$$\left.\begin{array}{l} \text{find } u = (u_1,\cdots,u_m) \text{ where } u_i \in L^2(0,T;\mathbf{R}^p) \text{ such that:} \\ \quad (1)\ z(x,t,u) \text{ is the solution of the system (4),} \\ \quad (2) \text{ the family } \left\{\omega_{t_i}^{u_i}\right\}_{0\leq i\leq m} \text{ is non-decreasing.} \end{array}\right\} \tag{15}$$

So the adaptive null-spreading control problem may be seen as a sequence of regional control problems, i.e., given $(t_i)_{0\leq i\leq m}$ and $(\omega_i)_{0\leq i\leq m}$, the problem becomes:

$$\left.\begin{array}{l} \text{find } u = (u_1,\cdots,u_m) \text{ where } u_i \in L^2(0,T;\mathbf{R}^p) \text{ such that:} \\ \quad (1)\ z(x,t,u) \text{ is the solution of the system (4),} \\ \quad (2)\ z(x,t_i,u_i) = z(x,t_i,u_i,z_0,t_0) = 0 \text{ on } \omega_i. \end{array}\right\} \tag{16}$$

Next, we try to solve problem (16) by considering the regional analysis theory.

3.2 Preliminaries of regional controllability

For any subregion $\omega \subseteq \Omega$ and $0 \leq t_i \leq T$, $i = 0,1,2,\cdots,m$, consider the following restriction mappings

χ_ω and its adjoint operator χ_ω^*

$$\chi_\omega:\ \begin{cases} L^2(\Omega)\to L^2(\omega), \\ \xi \to \xi|_\omega, \end{cases} \qquad \chi_\omega^* z(x) := \begin{cases} z(x), & x\in\omega, \\ 0, & x\in\Omega\backslash\omega, \end{cases}$$

we then see the following definition.

Definition 3.1 *The system* (4) *is said to be* $\omega-$*regionally controllable on the time interval* $[t_0,t_i]$ *if for any* $z_0 = z(t_0)\in\mathscr{D}(A)$ *and* $y_d\in L^2(\omega)$, *there exists a control* $u\in L^2(t_0,t_i;\mathbf{R}^p)$ *such that*

$$\chi_\omega z(x,t_i,u) = y_d, \tag{17}$$

where $z(x,t_i,u) = z(x,t_i,u,z_0,t_0)$ *is the solution of* (4) *at the final time* t_i *from the initial condition* z_0 *at time* $t=t_0$ *excited by the control input* u.

By [20, 27], we recall that the solution of the system (4) at time t_i from the initial condition z_0 at time $t=t_0$ excited by the control input u is given by

$$z(x,t_i,u) = t_i^{\alpha-1}K_\alpha(t_i)z_0 + G_{t_i}u,\ \ t_i\in I. \tag{18}$$

Then the relation (17) may be expressed as follows

$$\chi_\omega G_{t_i}u = y_d - \chi_\omega t_i^{\alpha-1}K_\alpha(t_i)z_0. \tag{19}$$

From the Proposition 3.1 in [31], the system (4) is $\omega-$regionally controllable on the time interval $[t_0,t_i]$ if and only if

$$Im\,(\chi_\omega G_{t_i}) = L^2(\omega), \tag{20}$$

where $Im(*)$ is the range of the operator $*$. Moreover, it should be pointed out that regional controllability as defined above is very strong. We usually consider the concepts of weak regional controllability, which is defined by $\overline{Im\,(\chi_\omega G_{t_i})} = L^2(\omega)$.

3.3 Solution of the adaptive spreading control problem

Given a sequence of increasing subregions $(\omega_i)_{1\le i\le m}$, $\omega_i\subseteq\Omega$, $i=0,1,\cdots,m$ and consider the following minimum energy control problem

$$\begin{cases} \inf\limits_u J(u) \\ J(u) := \int_{t_0}^{t_i}\|u(t)\|^2_{\mathbf{R}^p}dt,\ u\in U_T \text{ and} \\ U_T = \{u\in L^2(0,T;\mathbf{R}^p): \chi_\omega z(\cdot,t_i,u) = y_d\}. \end{cases} \tag{21}$$

where, obviously, U_T is a closed convex set. Then we see the following theorem.

Theorem 3.1 *If the system* (4) *is* $\omega-$*regionally controllable on the time interval* $[t_0,t_i]$, *then for any* $y_d\in L^2(\omega)$, *the minimum energy problem* (21) *has a unique solution* u^* *given by*

$$u^*(t) = (\chi_\omega G_{t_i})^* R_{t_i,\omega}^{-1}\left(y_d - \chi_\omega t_i^{\alpha-1}K_\alpha(t_i)z_0\right), \tag{22}$$

where $R_{t_i,\omega} = \chi_\omega G_{t_i}G_{t_i}^*\chi_\omega^*$.

Proof. To begin with, we claim that if the system (4) is $\omega-$regionally controllable on the time interval $[t_0,t_i]$, then

$$\|f\| := \int_{t_0}^{t_i}\|B^*(t_i-s)^{\alpha-1}K_\alpha^*(t_i-s)\chi_\omega^* f\|^2 ds \tag{23}$$

is a norm of space $L^2(\omega)$. In fact, if the system (4) is $\omega-$regionally controllable on $[t_0,t_i]$, we get that $Ker\left(G_{t_i}^*\chi_\omega^*\right) = \{0\}$, i.e., $B^*(t_i-s)^{\alpha-1}K_\alpha^*(t_i-s)\chi_\omega^* f = 0$ can implies $f=0$. Hence, for any $f\in L^2(\omega)$, it then follows from

$$\|f\| = 0 \ \Leftrightarrow\ B^*(t_i-s)^{\alpha-1}K_\alpha^*(t_i-s)\chi_\omega^* f = 0 \tag{24}$$

that $\|\cdot\|$ is a norm of space $L^2(\omega)$.

Moreover, we show that the operator $R_{t_i,\omega}$ is coercive. For any $y_1\in L^2(\omega)$, there exists a control $u\in L^2(t_0,t_i;\mathbf{R}^p)$ such that

$$y_1 = \chi_\omega\left[t_i^{\alpha-1}K_\alpha(t_i)z_0 + G_{t_i}u\right]. \tag{25}$$

It then follows that

$$\begin{aligned} &\langle R_{t_i,\omega}y_1, y_1\rangle_{L^2(\omega)} = \left\|G_{t_i}^*\chi_\omega^* y_1\right\|^2_{L^2(0,t_i,\mathbf{R}^p)} \\ &= \left\|B^*(t_i-\cdot)^{\alpha-1}K_\alpha^*(t_i-\cdot)\chi_\omega^* y_1\right\|^2_{L^2(0,t_i,\mathbf{R}^p)} \\ &\ge \|y_1\|^2. \end{aligned} \tag{26}$$

Moreover, since $R_{t_i,\omega}\in\mathscr{L}\left(L^2(\omega),L^2(\omega)\right)$, by the Theorem 1.1 in [32], we get that $R_{t_i,\omega}$ is coercive.

Next, since the solution of (4) excited by the control u^* is given by

$$z(x,t,u^*) = t^{\alpha-1}K_\alpha(t)z_0 + G_t u, \tag{27}$$

we have

$$\begin{aligned} \chi_\omega z(x,t_i,u^*) &= \chi_\omega\left[t_i^{\alpha-1}K_\alpha(t_i)z_0 + G_{t_i}u^*\right] \\ &= \chi_\omega t_i^{\alpha-1}K_\alpha(t_i)z_0 + \chi_\omega G_{t_i}u^* \\ &= y_d. \end{aligned} \tag{28}$$

Finally, we prove that u^* solves the minimum energy problem (21). For this purpose, since $\chi_\omega z(\cdot,t_i,u^*) = y_d$, for any $u\in L^2(0,t_i;\mathbf{R}^p)$ with $\chi_\omega z(\cdot,t_i,u) = y_d$, one has

$$\chi_\omega\left[z(\cdot,t_i,u^*) - z(\cdot,t_i,u)\right] = 0, \tag{29}$$

which follows that

$$\begin{aligned} 0 &= \chi_\omega\int_{t_0}^{t_i}(t_i-s)^{\alpha-1}K_\alpha(t_i-s)B[u^*(s)-u(s)]ds \\ &= \chi_\omega G_{t_i}[u^*-u]. \end{aligned}$$

Thus,

$$\begin{aligned} &J'(u^*)(u^*-u) \\ &= 2\int_{t_0}^{t_i}\langle u^*(s)-u(s), u^*(s)\rangle ds \\ &= 2\int_{t_0}^{t_i}\left\langle \begin{matrix} u^*(s)-u(s), (\chi_\omega G_{t_i})^* R_{t_i,\omega}^{-1}\times \\ \left(y_d - \chi_\omega t_i^{\alpha-1}K_\alpha(t_i)z_0\right) \end{matrix}\right\rangle ds \\ &= 2\int_{t_0}^{t_i}\left\langle \begin{matrix} \chi_\omega G_{t_i}[u^*(s)-u(s)], R_{t_i,\omega}^{-1}\times \\ \left(y_d - \chi_\omega t_i^{\alpha-1}K_\alpha(t_i)z_0\right) \end{matrix}\right\rangle ds \\ &= 0, \end{aligned}$$

it follows that $J(u) \geq J(u^*)$. Then by the Theorem 1.3 in [32], we see that u^* solves the minimum energy problem (21) and the proof is complete.

Theorem 3.2 *The adaptive null-spreading control problem* (16) *has at least one solution provided that the system* (4) *is weakly controllable on* $I = [0,T]$.

Proof. Suppose that $0 = t_0 < t_1 < \cdots < t_m = T$ is a given sequence of times and consider the zones

$$\omega_{t_i}^u = \{x \in \Omega : z(x,t_i,u) = 0\}, \tag{30}$$

$u \in L^2(0,T;\mathbf{R}^p)$ and $i = 1,2,\cdots,m$. Now if the system (4) is ω_0-regionally controllable, then by Theorem 3.1, the control

$$u_1^* = -\left(\chi_{\omega_0} G_{t_1}\right)^* R_{t_1,\omega_0}^{-1} \chi_{\omega_0} t_1^{\alpha-1} K_\alpha(t_1) z_0 \tag{31}$$

steers the system from z_0 to zero on ω_0, i.e., $\chi_{\omega_0} z(\cdot,t_1,u_1^*,z_0,t_0) = 0$.

Let $z_1(x) = z(x,t_1,u_1^*,z_0,t_0)$ and consider the set

$$\omega_1 = \{x \in \Omega : z_1(x) = 0\}, \tag{32}$$

by (14), we get that $\omega_0 \subseteq \omega_1$.

Next, we shall explore u_2^*, which steers (4) from z_1 at $t = t_1$ to zero at $t = t_2$ on ω_1, By [20], we see that

$$K_\alpha(t) z(x) = \sum_{j=1}^{\infty} E_{\alpha,\alpha}(\lambda_j t^\alpha)(z,\xi_j)\xi_j(x), \tag{33}$$

where $\{\lambda_j\}_{j\geq 1}$ and $\{\xi_j\}_{j\geq 1}$ are respectively, the eigenvalues and eigenvectors of operator A, and

$$E_{\alpha,\beta}(z) := \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\alpha i + \beta)}, \quad \mathbf{Re}\,\alpha > 0,\ \beta, z \in \mathbf{C} \tag{34}$$

is known as the generalized Mittag-Leffler function in two parameters. It then follows from the properties of Mittag-Leffler function that $K_\alpha^{-1}(t)$ exists [9, 10]. Assume that $u \equiv 0$ in system (4), we have

$$z(x,t,u) = t^{\alpha-1} K_\alpha(t) z_0, \quad t \in I. \tag{35}$$

Then $z_1 = t_1^{\alpha-1} K_\alpha(t_1) z_0^*$ and $z_0^* = t_1^{1-\alpha} K_\alpha^{-1}(t_1) z_1$. By Theorem 3.1, the control

$$u(t) = -\left[(\chi_{\omega_1} G_{t_2})^* R_{t_2,\omega_1}^{-1} \chi_{\omega_1} t_2^{\alpha-1} K_\alpha(t_2) z_0^*\right](t) \tag{36}$$

may steer the system from z_0^* at $t = t_0$ to zero at $t = t_2$ on ω_1. Then we get that the controller

$$u_2^* = -\frac{t_1^{1-\alpha}}{t_2^{1-\alpha}} (\chi_{\omega_1} G_{t_2})^* R_{t_2,\omega_1}^{-1} \chi_{\omega_1} \frac{K_\alpha(t_2)}{K_\alpha(t_1)} z_1 \tag{37}$$

can excite the system to zero at $t = t_2$ on ω_1.

Following the same procedure, let $z_{i-1}(x) = z(x,t_{i-1},u_{i-1}^*,z_{i-2},t_{i-2})$, we see that the controller

$$u_i^* = -(t_{i-1}/t_i)^{1-\alpha} \left(\chi_{\omega_{i-1}} G_{t_i}\right)^* R_{t_i,\omega_{i-1}}^{-1} \chi_{\omega_{i-1}} K_\alpha(t_i) K_\alpha^{-1}(t_{i-1}) z_{i-1}. \tag{38}$$

could steer the system to zero at $t = t_i$ on ω_{i-1}. Now consider the control $u^* = (u_1^*, u_2^*, \cdots, u_m^*)$, it is easily to see that u^* solves the adaptive spreading control problems. The proof is complete.

4 Conclusion

This paper deals with the spreadability of the Riemann-Liouville time fractional diffusion systems of order $\alpha \in (0,1)$. An approach to guarantee the existence of solution to the adaptive spreading control based on the regional control results is proposed. At the same time, we introduce a new equivalence of the solution for the system (4) to support the proof of Theorem 3.2. The applications of those theoretical results are rich in our life. The results here provide some insights into the control theory of the design of fractional order diffusion equations, which can also be extended to complex fractional order distributed parameter dynamic systems. Various open questions are still under consideration. The problem of how many actuators/sensors are sufficient as well as how to best configure them to control/observe the spread process are of great interest.

References

[1] E. W. Montroll and G. H. Weiss, "Random walks on lattices. II," *J. Math. Phys.*, vol. 6, no. 2, pp. 167–181, 1965.

[2] A. Cartea and D. Negrete, "Fluid limit of the continuous-time random walk with general Lévy jump distribution functions," *Phys. Rev. E*, vol. 76, no. 4, pp. 1–8, 2007.

[3] G. Gradenigo, A. Sarracino, D. Villamaina, and A. Vulpiani, "Einstein relation in systems with anomalous diffusion," *Acta Phys. Polo. B*, vol. 44, no. 5, pp. 899–912, 2013.

[4] A. Sharifi-Viand, M. Mahjani, and M. Jafarian, "Investigation of anomalous diffusion and multifractal dimensions in polypyrrole film," *J. Electroanal. Chem.*, vol. 671, pp. 51–57, 2012.

[5] V. Uchaikin and R. Sibatov, "Fractional kinetics in solids: Anomalous charge transport in semiconductors," *Dielectrics and Nanosystems, World Science*, 2012.

[6] W. M. Spears and D. F. Spears, *Physicomimetics: Physics-based swarm intelligence*. Springer Science & Business Media, 2012.

[7] P. J. Torvik and R. L. Bagley, "On the appearance of the fractional derivative in the behavior of real materials," *J. Appl. Mech.*, vol. 51, no. 2, pp. 294–298, 1984.

[8] B. B. Mandelbrot, *The fractal geometry of nature*. Macmillan, 1983.

[9] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin, *Mittag-Leffler functions, related topics and applications*. Springer, 2014.

[10] A. M. Mathai and H. J. Haubold, *Special functions for applied scientists*. Springer, 2008.

[11] H. J. Haubold, A. M. Mathai, and R. K. Saxena, "Mittag-leffler functions and their applications," *J. Appl. Math.*, 2011.

[12] F. Mainardi and R. Gorenflo, "On Mittag-Leffler-type functions in fractional evolution processes," *J. Comput. Appl. Math.*, vol. 118, no. 1, pp. 283–299, 2000.

[13] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: a fractional dynamics approach," *Phys. rep.*, vol. 339, no. 1, pp. 1–77, 2000.

[14] M. M. Meerschaert and E. Scalas, "Coupled continuous time random walks in finance," *Phys. A*, vol. 370, no. 1, pp. 114–118, 2006.

[15] A. E. JAI and K. Kassara, "Spreadability of transport systems," *Internat. J. Systems Sci.*, vol. 27, no. 7, pp. 681–688, 1996.

[16] A. El Jai and K. Kassara, "Spreadable distributed systems," *Math. Comput. Modelling*, vol. 20, no. 1, pp. 47–64, 1994.

[17] A. El Jai, K. Kassara, and O. Cabrera, "Spray control," *Internat. J. Control*, vol. 68, no. 4, pp. 709–730, 1997.

[18] S. El Yacoubi, A. El Jai, and J. Karrakchou, "Spreadability and spray actuators," *Appl. Math. Comput. Sci.*, vol. 8, pp. 367–379, 1998.

[19] K. Kassara, "Feedback spreading control under speed constraints," *SIAM J. Control Optim.*, vol. 41, no. 4, pp. 1281–1294, 2002.

[20] F. Ge, Y. Chen, and C. Kou, "Regional boundary controllability of time fractional diffusion processes," *IMA J. Math. Control Inform.*, pp. 1–18, 2016.

[21] ——, "Regional controllability of anomalous diffusion generated by the time fractional diffusion equations," *ASME IDETC/CIE 2015, Boston, Aug. 2-5, 2015. See also: arXiv:1508.00047*, 2015.

[22] I. Podlubny, *Fractional differential equations*. Academic press, 1999.

[23] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*. Elsevier Science Limited, 2006.

[24] A. Bernoussi and A. El Jai, "New approach of spreadability," *Math. Comput. Modelling*, vol. 31, no. 13, pp. 93–109, 2000.

[25] J. P. Aubin, *Viability theory*. Springer Science & Business Media, 2009.

[26] A. El Jai, M. Fournier, K. Kassara, and B. Noumare, "Vegetation dynamics: A deterministic modelling approach using the concept of spreadability," *Theor. Appl. Climatol.*, vol. 52, no. 3-4, pp. 241–249, 1995.

[27] Z. Liu and X. Li, "Approximate controllability of fractional evolution systems with Riemann–Liouville fractional derivatives," *SIAM J. Control Optim*, vol. 53, no. 4, pp. 1920–1933, 2015.

[28] F. Mainardi, P. Paradisi, and R. Gorenflo, "Probability distributions generated by fractional diffusion equations," *2007. arXiv preprint arXiv:0704.0320*.

[29] Y. Zhou and F. Jiao, "Existence of mild solutions for fractional neutral evolution equations," *Comput. Math. Appl.*, vol. 59, no. 3, pp. 1063–1077, 2010.

[30] D. Uciński and A. El Jai, "On weak spreadability of distributed-parameter systems and its achievement via linear–quadratic control techniques," *IMA J. Math. Control Inform.*, vol. 14, no. 2, pp. 153–174, 1997.

[31] F. Ge, Y. Chen, and C. Kou, "Actuator characterisations to achieve approximate controllability for a class of fractional sub-diffusion equations," *Internat. J. Control*, pp. 1–9, 2016.

[32] J. L. Lions, *Optimal control of systems governed by partial differential equations*. Springer Verlag, 1971.