

Regional controllability analysis of fractional diffusion equations with Riemann-Liouville time fractional derivatives

Fudong Ge ^{a,b}, YangQuan Chen ^{c,†}, Chunhai Kou ^d

^a*School of Computer Science, China University of Geosciences, Wuhan 430074, PR China*

^b*Hubei Key Laboratory of Intelligent Geo-Information Processing, China University of Geosciences, Wuhan 430074, PR China*

^c*Mechatronics, Embedded Systems and Automation Lab, University of California, Merced, CA 95343, USA*

^d*Department of Applied Mathematics, Donghua University, Shanghai 201620, PR China*

Abstract

This paper is concerned with the concepts of regional controllability for the Riemann-Liouville time fractional diffusion systems of order $\alpha \in (0, 1)$. The characterizations of strategic actuators to achieve regional controllability are investigated when the control inputs emerge in the differential equations as distributed inputs. In the end, an approach to guarantee the regional controllability of the problems under consideration in the considered subregion with minimum energy control is described and successfully tested through two applications.

Key words: Regional controllability; Time fractional diffusion systems; Strategic actuators; Minimum energy control.

1 Introduction

Recently sub-diffusion processes have attracted increasing interest since the introduction of continuous time random walks (CTRWs) in [Montroll & Weiss, 1965] and a large number of contributions have been given to them ([Mainardi et al., 2007, Metzler & Klafter, 2000], [Ge et al., 2016b, Fujishiro & Yamamoto, 2014]). Since CTRW is a random walk subordinated to a simple renewal process, by [Hilfer & Anton, 1995], it can be regarded as a generalized physical diffusion process (including the sub-diffusion process and the super-diffusion process) and there exists a closed connection between the time fractional diffusion system and the sub-diffusion process. Moreover, it is confirmed in [Metzler & Klafter, 2000] and [Mandelbrot, 1983] that the time fractional diffusion systems can be used to well

characterize those sub-diffusion processes, which offer better performance not achievable before using conventional diffusion systems and surely raise many potential research opportunities at the same time.

In the case of diffusion system, it is well known that in general, not all the states can be reached in the whole domain of interest. So here, we first introduce some notations on the regional controllability of time fractional diffusion systems when the system under consideration is only exactly (or approximately) controllability on a subset of the whole space, which can be regarded as an extensions of the research work in ([El Jai et al., 1995], [Sakawa, 1974]). Besides, focusing on regional controllability would allow for a reduction in the number of physical actuators, offer the potential to reduce computational requirements in some cases, and also possible to discuss those systems which are not controllable on the whole domain, etc.

Furthermore, in [Chen & Feng, 2016, Ge et al., 2016a] and [El Jai & Pritchard, 1988], the authors have shown that the measurements and actions in practical systems can be better described by using the notion of actuators and sensors (including the location, number and spatial distribution of actuators and sensors [El Jai, 1991]). Then the contribution of this present work is on the

* This work was completed while F. Ge visited the Mechatronics, Embedded Systems and Automation Lab, University of California, Merced from October 2014 to October 2015.

†Corresponding author. Tel./Fax:+1(209)228-4672/4047.

Email addresses: gefd2011@gmail.com (Fudong Ge), yqchen@ieee.org (YangQuan Chen), kouchunhai@dhu.edu.cn (Chunhai Kou).

regional controllability of the sub-diffusion processes described by Riemann-Liouville time fractional diffusion systems of order $\alpha \in (0, 1)$ by using the notion of actuators and sensors. As cited in [Hilfer, 2000], their applications are rich in many real life. For example, the flow through porous media ([Uchaikin & Sibatov, 2012]), or the swarm of robots moving through dense forest ([Spears & Spears, 2012]). We hope that the results here could provide some insights into the qualitative analysis of the design and configuration of fractional controller.

The rest of the paper is organized as follows. The mathematical concept of regional controllability problem is presented in the next section. Section 3 is focused on the characterizations of strategic actuators in the case of regional controllability. In Section 4, our main results on the regional controllability analysis of time fractional diffusion systems are presented and the determination of the optimal control which achieves the regional controllability is obtained. Two applications are worked out in the last section.

2 Statement of the problem

Let Ω be an open bounded subset of \mathbf{R}^n with smooth boundary $\partial\Omega$ and we consider the following abstract Riemann-Liouville time fractional differential system:

$$\begin{cases} {}_0D_t^\alpha z(t) = Az(t) + Bu(t), & t \in [0, b], \quad 0 < \alpha < 1, \\ \lim_{t \rightarrow 0^+} {}_0D_t^\alpha z(t) = z_0, \end{cases} \quad (1)$$

where A generates a strongly continuous semigroup $\{\Phi(t)\}_{t \geq 0}$ on the Hilbert space $Z := L^2(\Omega)$, $-A$ is a uniformly elliptic operator ([Renardy & Rogers, 2006], [Weinberger, 1962]), $z \in L^2(0, b; Z)$ and the initial vector $z_0 \in Z$. Here ${}_0D_t^\alpha$ and ${}_0I_t^\alpha$ denote the Riemann-Liouville fractional order derivative and integral, respectively, given by [Kilbas et al., 2006]

$${}_0D_t^\alpha z(t) = \frac{d}{dt} {}_0I_t^{1-\alpha} z(t), \quad 0 < \alpha < 1 \quad (2)$$

and

$${}_0I_t^\alpha z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, \quad \alpha > 0. \quad (3)$$

In addition, B is a control operator depends on the number and the structure of actuators. The control $u \in U$ where U is a Hilbert space. In particular, if the system is excited by p actuators, one has $u \in L^2(0, b; \mathbf{R}^p)$ and $B : \mathbf{R}^p \rightarrow Z$.

We first recall some necessary lemmas to be used afterwards.

Lemma 1 For any given $f \in L^2(0, b; Z)$, $0 < \alpha < 1$, a function $v \in L^2(0, b; Z)$ is said to be a mild solution of the following system

$$\begin{cases} {}_0D_t^\alpha v(t) = Av(t) + f(t), & t \in [0, b], \\ \lim_{t \rightarrow 0^+} {}_0D_t^{\alpha-1} v(t) = v_0 \in Z, \end{cases} \quad (4)$$

if it satisfies

$$v(t) = t^{\alpha-1} K_\alpha(t) v_0 + \int_0^t (t-s)^{\alpha-1} K_\alpha(t-s) f(s) ds, \quad (5)$$

where

$$K_\alpha(t) = \alpha \int_0^\infty \theta \phi_\alpha(\theta) \Phi(t^\alpha \theta) d\theta. \quad (6)$$

Here $\{\Phi(t)\}_{t \geq 0}$ is the strongly continuous semigroup generated by operator A , $\phi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \psi_\alpha(\theta^{-\frac{1}{\alpha}})$ and ψ_α is a probability density function defined by ($\theta > 0$)

$$\psi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-\alpha n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha) \quad (7)$$

such that ([Mainardi et al., 2007])

$$\int_0^\infty \psi_\alpha(\theta) d\theta = 1 \quad \text{and} \quad \int_0^\infty \theta^\nu \phi_\alpha(\theta) d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}, \quad \nu \geq 0.$$

Proof. It follows from the Laplace transforms

$$\tilde{v}(\lambda) = \int_0^\infty e^{-\lambda s} v(s) ds \quad \text{and} \quad \tilde{f}(\lambda) = \int_0^\infty e^{-\lambda s} f(s) ds \quad (8)$$

that the system (1) is equivalent to ([Lin & Lu, 2013])

$$\lambda^\alpha \tilde{v}(\lambda) - v_0 - A\tilde{v}(\lambda) = \tilde{f}(\lambda). \quad (9)$$

Then

$$\begin{aligned} \tilde{v}(\lambda) &= (\lambda^\alpha I - A)^{-1} (v_0 + \tilde{f}(\lambda)) \\ &= \int_0^\infty e^{-\lambda^\alpha s} \Phi(s) [v_0 + \tilde{f}(\lambda)] ds. \end{aligned} \quad (10)$$

Consider the stable probability density function (7). By the arguments in [Mainardi et al., 2007], we see that $\psi_\alpha(\theta)$ ($\theta > 0$) satisfies the following property

$$\tilde{\psi}_\alpha(\lambda) = \int_0^\infty e^{-\lambda \theta} \psi_\alpha(\theta) d\theta = e^{-\lambda^\alpha}, \quad \alpha \in (0, 1). \quad (11)$$

Let $s = \tau^\alpha$. We obtain that

$$\begin{aligned}\tilde{v}(\lambda) &= \alpha \int_0^\infty e^{-\lambda^\alpha \tau^\alpha} \Phi(\tau^\alpha) \tau^{\alpha-1} [v_0 + \tilde{f}(\lambda)] d\tau \\ &= \alpha \int_0^\infty \int_0^\infty e^{-\lambda \tau^\theta} \psi_\alpha(\theta) \Phi(\tau^\alpha) \tau^{\alpha-1} [v_0 + \tilde{f}(\lambda)] d\theta d\tau \\ &= \sigma_1(v_0) + \sigma_2(f),\end{aligned}$$

where $\sigma_1(v_0) = \alpha \int_0^\infty \int_0^\infty e^{-\lambda \tau^\theta} \psi_\alpha(\theta) \Phi(\tau^\alpha) \tau^{\alpha-1} d\theta d\tau v_0$ and $\sigma_2(f) = \alpha \int_0^\infty \int_0^\infty e^{-\lambda \tau^\theta} \psi_\alpha(\theta) \Phi(\tau^\alpha) \tau^{\alpha-1} \tilde{f}(\lambda) d\theta d\tau$. Suppose that $t = \tau^\theta$. Then we have

$$\begin{aligned}\sigma_1(v_0) &= \alpha \int_0^\infty \int_0^\infty e^{-\lambda t} \psi_\alpha(\theta) \Phi\left(\frac{t^\alpha}{\theta^\alpha}\right) \frac{t^{\alpha-1}}{\theta^\alpha} d\theta dt v_0 \\ &= \int_0^\infty e^{-\lambda t} \alpha \int_0^\infty \psi_\alpha(\theta) \Phi(t^\alpha \theta^{-\alpha}) t^{\alpha-1} \theta^{-\alpha} d\theta dt v_0 \\ &= \int_0^\infty e^{-\lambda t} t^{\alpha-1} \alpha \int_0^\infty \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \psi_\alpha(\theta^{-\frac{1}{\alpha}}) \theta \Phi(t^\alpha \theta) d\theta dt v_0\end{aligned}$$

and

$$\begin{aligned}\sigma_2(f) &= \alpha \int_0^\infty \int_0^\infty \int_0^\infty e^{-\lambda \tau^\theta} \psi_\alpha(\theta) \Phi(\tau^\alpha) \tau^{\alpha-1} e^{-\lambda s} f(s) ds d\theta d\tau \\ &= \alpha \int_0^\infty \int_0^\infty \int_0^\infty e^{-\lambda(t+s)} \psi_\alpha(\theta) \Phi\left(\frac{t^\alpha}{\theta^\alpha}\right) \frac{t^{\alpha-1}}{\theta^\alpha} f(s) ds d\theta dt \\ &= \int_0^\infty e^{-\lambda t} \alpha \int_0^t \int_0^\infty \psi_\alpha(\theta) \Phi\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right) \frac{(t-s)^{\alpha-1} f(s)}{\theta^\alpha} d\theta ds dt \\ &= \int_0^\infty e^{-\lambda t} \alpha \int_0^t \int_0^\infty \theta \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \psi_\alpha(\theta^{-\frac{1}{\alpha}}) \frac{\Phi((t-s)^\alpha \theta) f(s)}{(t-s)^{1-\alpha}} d\theta ds dt.\end{aligned}$$

Let $\phi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \psi_\alpha(\theta^{-\frac{1}{\alpha}})$ and $K_\alpha(t) = \alpha \int_0^\infty \theta \phi_\alpha(\theta) \Phi(t^\alpha \theta) d\theta$. Then we get

$$v(t) = t^{\alpha-1} K_\alpha(t) v_0 + \int_0^t (t-s)^{\alpha-1} K_\alpha(t-s) f(s) ds$$

and the proof is complete.

Lemma 2 [Dacorogna, 2007] *Let $\Omega \subseteq \mathbf{R}^n$ be an open set and $C_0^\infty(\Omega)$ be the class of infinitely differentiable functions on Ω with compact support in Ω and $u \in L_{loc}^1(\Omega)$ be such that*

$$\int_\Omega u(x) \psi(x) dx = 0, \quad \forall \psi \in C_0^\infty(\Omega). \quad (12)$$

Then $u = 0$ almost everywhere in Ω .

Lemma 3 [Klimek, 2009] *Let the reflection operator Q on interval $[0, b]$ be as follows:*

$$Qf(t) := f(b-t). \quad (13)$$

Then the following equations hold:

$$Q_0 I_t^\alpha f(t) = {}_t I_b^\alpha Qf(t), \quad Q_0 D_t^\alpha f(t) = {}_t D_b^\alpha Qf(t) \quad (14)$$

and

$${}_0 I_t^\alpha Qf(t) = Q {}_t I_b^\alpha f(t), \quad {}_0 D_t^\alpha Qf(t) = Q {}_t D_b^\alpha f(t). \quad (15)$$

Let $\omega \subseteq \Omega$ be a given region of positive Lebesgue measure and $z_b \in L^2(\omega)$ (the target function) be a given element. By Lemma 1, the unique mild solution $z(\cdot, u)$ of (1) can be given by

$$z(t, u) = t^{\alpha-1} K_\alpha(t) z_0 + \int_0^t (t-s)^{\alpha-1} K_\alpha(t-s) B u(s) ds.$$

Taking into account that (1) is a line system, by the Proposition 3.1 in [Ge et al., 2016a], it suffices to suppose that $z_0 = 0$ in the following discussion. Let $H : L^2(0, b; \mathbf{R}^p) \rightarrow Z$ be

$$H u = \int_0^b \frac{K_\alpha(b-s)}{(b-s)^{1-\alpha}} B u(s) ds, \quad \forall u \in L^2(0, b; \mathbf{R}^p). \quad (16)$$

In order to state the main results, the following two assumptions are supposed to hold all over the article:

(A₁) B is a densely defined operator and B^* exists.

(A₂) $(BK_\alpha(t))^*$ exists and $(BK_\alpha(t))^* = K_\alpha^*(t)B^*$.

In particular, when $B \in \mathcal{L}(\mathbf{R}^p, Z)$ is a bounded linear operator from \mathbf{R}^p to Z , it is easy to see that (A₁) and (A₂) hold. Suppose that $\{\Phi^*(t)\}_{t \geq 0}$, generated by the adjoint operator of A , is also a strongly continuous semigroup in the space Z . For any $v \in L^2(\Omega)$, by $\langle H u, v \rangle = \langle u, H^* v \rangle$, we have

$$H^* v = B^*(b-s)^{\alpha-1} K_\alpha^*(b-s) v, \quad (17)$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing of the space Z , B^* is the adjoint operator of B and $K_\alpha^*(t) = \alpha \int_0^\infty \theta \phi_\alpha(\theta) \Phi^*(t^\alpha \theta) d\theta$. Consider now the restriction map

$$p_\omega : L^2(\Omega) \rightarrow L^2(\omega), \quad (18)$$

defined by $p_\omega z = z|_\omega$, is the projection operator on ω . Then the adjoint operator of p_ω can be given by

$$p_\omega^* z(x) := \begin{cases} z(x), & x \in \omega, \\ 0, & x \in \Omega \setminus \omega. \end{cases} \quad (19)$$

and we are ready to state the following definition.

Definition 4 (i) *The system (1) is said to be regionally exactly controllable on ω if for any $z_b \in L^2(\omega)$ at time b ,*

(2) For each $j = 1, 2, \dots$, ξ_{jk} ($k = 1, 2, \dots, r_j$) is the orthonormal eigenfunction corresponding to λ_j , i.e.,

$$(\xi_{jk_m}, \xi_{jk_n}) = \begin{cases} 1, & k_m = k_n, \\ 0, & k_m \neq k_n, \end{cases}$$

where $1 \leq k_m, k_n \leq r_j$, $k_m, k_n \in \mathbf{N}$ and (\cdot, \cdot) is the inner product of space $L^2(\Omega)$.

Then we see that the strongly continuous semigroup $\{\Phi(t)\}_{t \geq 0}$ on Z generated by A is

$$\Phi(t)z(x) = \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} \exp(-\lambda_j t)(z, \xi_{jk}) \xi_{jk}(x), \quad x \in \Omega \quad (24)$$

and the sequence $\{\xi_{jk}, k = 1, 2, \dots, r_j, j = 1, 2, \dots\}$ is an orthonormal basis in $L^2(\Omega)$, then for any $z(x) \in L^2(\Omega)$, it can be expressed as

$$z(x) = \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} (z, \xi_{jk}) \xi_{jk}(x).$$

Definition 8 An actuators (or a suite of actuators) is said to be ω -strategic if the system under consideration is regionally approximately controllable on ω .

Before showing our main result in this part, from Eq.(6) and Eq.(24), for any $z \in L^2(\Omega)$, we have

$$\begin{aligned} & K_{\alpha}(t)z(x) \\ &= \alpha \int_0^{\infty} \theta \phi_{\alpha}(\theta) \Phi(t^{\alpha} \theta) z(x) d\theta \\ &= \alpha \int_0^{\infty} \theta \phi_{\alpha}(\theta) \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} \exp(-\lambda_j t^{\alpha} \theta) (z, \xi_{jk}) \xi_{jk}(x) d\theta \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} \sum_{n=0}^{\infty} \frac{\alpha (-\lambda_j t^{\alpha})^n}{n!} (z, \xi_{jk}) \xi_{jk}(x) \int_0^{\infty} \theta^{n+1} \phi_{\alpha}(\theta) d\theta \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} \sum_{n=0}^{\infty} \frac{\alpha (n+1)! (-\lambda_j t^{\alpha})^n}{\Gamma(\alpha n + \alpha + 1) n!} (z, \xi_{jk}) \xi_{jk}(x) \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} \alpha E_{\alpha, \alpha+1}^2(-\lambda_j t^{\alpha}) (z, \xi_{jk}) \xi_{jk}(x), \end{aligned}$$

where $E_{\alpha, \beta}^{\mu}(z) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}$, $z \in \mathbf{C}$, $\alpha, \beta, \mu \in \mathbf{C}$,

$\mathbf{Re} \alpha > 0$ is the generalized Mittag-Leffler function in three parameters and here, $(\mu)_n$ is the Pochhammer symbol defined by (see [Erdélyi et al., 1953], Section 2.1.1)

$$(\mu)_n = \mu(\mu+1) \cdots (\mu+n-1), \quad n \in \mathbf{N}. \quad (25)$$

Moreover, If $\alpha, \beta \in \mathbf{C}$ such that $\mathbf{Re} \alpha > 0$, $\mathbf{Re} \beta > 1$, then (see Section 2.3.4 in [Mathai & Haubold, 2008], or Section 5.1.1 in [Gorenflo et al., 2014])

$$\alpha E_{\alpha, \beta}^2 = E_{\alpha, \beta-1} - (1 + \alpha - \beta) E_{\alpha, \beta}. \quad (26)$$

It follows that

$$K_{\alpha}(t)z(x) = \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} E_{\alpha, \alpha}(-\lambda_j t^{\alpha})(z, \xi_{jk}) \xi_{jk}(x) \quad (27)$$

and

$$\begin{aligned} & \int_0^t \tau^{\alpha-1} K_{\alpha}(\tau) B u(t-\tau) d\tau \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} \sum_{i=1}^p \int_0^t g_{jk}^i u_i(t-\tau)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_j \tau^{\alpha}) d\tau \xi_{jk}(x), \end{aligned}$$

where $E_{\alpha, \beta}(z) := \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\alpha i + \beta)}$, $\mathbf{Re} \alpha > 0$, $\beta, z \in \mathbf{C}$ is

known as the generalized Mittag-Leffler function in two parameters and $g_{jk}^i = (p_{D_i} g_i, \xi_{jk})$, $j = 1, 2, \dots$, $k = 1, 2, \dots, r_j$, $i = 1, 2, \dots, p$. Then we have the following result.

Theorem 9 For $j = 1, 2, \dots$, arbitrary given $b > 0$, define $p \times r_j$ matrices G_j as

$$G_j = \begin{bmatrix} g_{j1}^1 & g_{j2}^1 & \cdots & g_{jr_j}^1 \\ g_{j1}^2 & g_{j2}^2 & \cdots & g_{jr_j}^2 \\ \vdots & \vdots & \vdots & \vdots \\ g_{j1}^p & g_{j2}^p & \cdots & g_{jr_j}^p \end{bmatrix}_{p \times r_j}, \quad (28)$$

where $g_{jk}^i = (p_{D_i} g_i, \xi_{jk})$, $j = 1, 2, \dots$, $k = 1, 2, \dots, r_j$, $i = 1, 2, \dots, p$. Then the suite of actuators $(D_i, g_i)_{1 \leq i \leq p}$ is said to be ω -strategic if and only if

$$p \geq r = \max\{r_j\} \text{ and rank } G_j = r_j, \quad j = 1, 2, \dots. \quad (29)$$

Proof. For any given $b > 0$ and all $u \in L^2(0, b; \mathbf{R}^p)$, suppose that $z_* \in L^2(\omega)$ satisfies

$$\begin{aligned} & (p_{\omega} H u, z_*) \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} \sum_{i=1}^p \int_0^b \frac{E_{\alpha, \alpha}(-\lambda_j \tau^{\alpha})}{\tau^{1-\alpha}} u_i(b-\tau) d\tau g_{jk}^i z_{jk} = 0, \end{aligned} \quad (30)$$

where $z_{jk} = (\xi_{jk}, z_*)_{L^2(\omega)}$, $j = 1, 2, \dots$, $k = 1, 2, \dots, r_j$. Moreover, since $u = (u_1, u_2, \dots, u_p)$ in (30) is arbitrary,

Lemma 2 leads us to

$$\sum_{j=1}^{\infty} \sum_{k=1}^{r_j} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^\alpha) g_{jk}^i z_{jk} = 0. \quad (31)$$

Then we see that the suite of actuators $(D_i, g_i)_{1 \leq i \leq p}$ is ω -strategic if and only if for any $z_* \in L^2(\omega)$, one has

$$\sum_{j=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^\alpha) G_j z_j = \mathbf{0}, \quad i = 1, 2, \dots, p, t \in [0, b]$$

$\Rightarrow z_* = \mathbf{0}$, where $\mathbf{0} = (0, 0, \dots, 0) \in \mathbf{R}^p$, $z_j = (z_{j1}, z_{j2}, \dots, z_{jr_j})^T$ is a vector in \mathbf{R}^{r_j} and $j = 1, 2, \dots$.

Finally, since $t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^\alpha) > 0$ for all $t \geq 0$, $j = 1, 2, \dots$, we then show our proof by using the Reductio and absurdum.

a) If the actuators $(D_i, g_i)_{1 \leq i \leq p}$ are not ω -strategic, i.e., the system (1) is not regionally approximately controllable on ω . There exists a $z_{j^*k} \neq 0$ satisfying

$$G_{j^*} z_{j^*} = \mathbf{0}. \quad (32)$$

Then if $p \geq r = \max\{r_j\}$, we see that

$$\text{rank } G_{j^*} < r_{j^*}. \quad (33)$$

b) On the contrary, if $p \geq r = \max\{r_j\}$ and $\text{rank } G_j < r_j$ for some $j = 1, 2, \dots$, there exists a nonzero element $\tilde{z} \in L^2(\omega)$ with $\tilde{z}_j = (\tilde{z}_{j1}, \tilde{z}_{j2}, \dots, \tilde{z}_{jr_j})^T \in \mathbf{R}^{r_j}$ such that

$$G_j \tilde{z}_j = \mathbf{0}. \quad (34)$$

Then there exists a nonzero element $\tilde{z} \in L^2(\omega)$ satisfying

$$\sum_{j=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^\alpha) G_j \tilde{z}_j = \mathbf{0}, \quad t \geq 0. \quad (35)$$

This implies that $\overline{\text{imp}_\omega H} \neq L^2(\omega)$ and the suite of actuators $(D_i, g_i)_{1 \leq i \leq p}$ is not ω -strategic. The proof is complete.

Remark 10 1) The system (1) with $\alpha = 1$,

$$A = - \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2} \right) + q$$

and $q(x)$ being Hölder continuous on the compact domain of \mathbf{R}^n is discussed in [Sakawa, 1974], which can be considered as a particular case of our results.

2) If the multiplicity of the eigenvalues λ_j of the operator $-A$ is infinite for some $j = 1, 2, \dots$ and if the system (1)

is regionally approximately controllable, then the number of the control functions should not be finite.

4 An approach for regional target control

The purpose of this section is to present an approach on how to achieve the regional approximate controllability on ω with the minimum control energy to steer the system (1) from the initial vector z_0 to a target function z_b in the region ω . The method used here is the Hilbert uniqueness methods (HUMs)[Lions, 1971].

Let U_b be the closed convex set defined by

$$U_b = \{u \in L^2(0, b; \mathbf{R}^p) : p_\omega z(b, u) = z_b\}. \quad (36)$$

Consider the following minimization problem

$$\inf_u J(u) = \inf_u \left\{ \int_0^b \|u(t)\|_{\mathbf{R}^p}^2 dt : u \in U_b \right\}. \quad (37)$$

Next, we show a direct approach to the solution of the regional controllability problem with minimum control energy by utilizing the HUMs.

Let G and E be the sets given by

$$G = \{g \in L^2(\Omega) : g = \mathbf{0} \text{ in } \Omega \setminus \omega\} \quad (38)$$

and

$$E = \{e \in L^2(\Omega) : e = \mathbf{0} \text{ in } \omega\}. \quad (39)$$

Then for $(g, e) \in G \times E$, we have

$$(g, e) = \int_\Omega g e dx = \int_\omega g e dx + \int_{\Omega \setminus \omega} g e dx = 0. \quad (40)$$

Moreover, for any $g \in G$, consider the system

$$\begin{cases} Q_t D_b^\alpha \varphi(t) = A^* Q \varphi(t), & t \in [0, b], \\ \lim_{t \rightarrow 0^+} Q_t D_b^{\alpha-1} \varphi(t) = p_\omega^* g \end{cases} \quad (41)$$

and the semi-norm on G

$$g \in G \rightarrow \|g\|_G^2 = \int_0^b \|B^* \varphi(t)\|^2 dt, \quad (42)$$

where the reflective operator Q is defined in (13).

Lemma 11 (42) defines a norm on G if the system (1) is regionally approximately controllable on ω .

Proof. For any $g \in G$, by Lemma 3, we see that system (41) can be rewritten as

$$\begin{cases} {}_0D_t^\alpha Q\varphi(t) = A^*Q\varphi(t), & t \in [0, b], \\ \lim_{t \rightarrow 0^+} {}_0D_t^{\alpha-1}Q\varphi(t) = p_\omega^*g \end{cases} \quad (43)$$

and its unique mild solution is

$$\varphi(t) = (b-t)^{\alpha-1}K_\alpha^*(b-t)p_\omega^*g. \quad (44)$$

Moreover, if the system (1) is regionally approximately controllable on ω , we have

$$\ker H^*p_\omega^* = \{0\}, \quad (45)$$

i.e.,

$$B^*(b-s)^{\alpha-1}K_\alpha^*(b-t)p_\omega^*g = 0 \Rightarrow g = 0. \quad (46)$$

Hence, for any $g \in G$, it follows from

$$\begin{aligned} \|g\|_G^2 &= \int_0^b \|B^*K_\alpha^*(b-s)p_\omega^*g\|^2 ds = 0 \\ \Leftrightarrow & B^*(b-s)^{\alpha-1}K_\alpha^*(b-s)p_\omega^*g = \mathbf{0} \end{aligned}$$

that $\|\cdot\|_G$ is a norm of space G and the proof is complete.

In addition, consider the following system

$$\begin{cases} {}_0D_t^\alpha \psi(t) = A\psi(t) + BB^*\varphi(t), & t \in [0, b], \\ \lim_{t \rightarrow 0^+} {}_0D_t^{\alpha-1}\psi(t) = 0, \end{cases} \quad (47)$$

which is controlled by the solution of the system (41). Let $\Lambda: G \rightarrow E^\perp$ be

$$\Lambda_g = p_\omega \psi(b). \quad (48)$$

Suppose that $\tilde{\psi}(t)$ satisfies

$$\begin{cases} {}_0D_t^\alpha \tilde{\psi}(t) = A\tilde{\psi}(t), \\ \lim_{t \rightarrow 0^+} \tilde{\psi}(t) = z_0. \end{cases} \quad (49)$$

For all $z_b \in L^2(\omega)$, we see that $z_b = p_\omega [\psi(b) + \tilde{\psi}(b)]$ and the regional controllability problem is equivalent to solving the equation

$$\Lambda_g := z_b - p_\omega \tilde{\psi}(b), \quad (50)$$

Then we can obtain the following theorem.

Theorem 12 *If the system (1) is regionally approximately controllable on ω , then for any $z_b \in L^2(\omega)$, (50) has a unique solution $g \in G$ and the control*

$$u^*(t) = B^*\varphi(t) \quad (51)$$

steers the system (1) to z_b at time b in ω . Moreover, u^ solves the minimum problem (37).*

Proof. By Lemma 11, we see that if the system (1) is regionally approximately controllable on ω , then $\|\cdot\|_G$ is a norm of space G . Let the completion of G with respect to the norm $\|\cdot\|_G$ again by G . Then we will show that (50) has a unique solution in G .

For any $g \in G$, it follows from the definition of operator Λ in (48) that

$$\begin{aligned} \langle g, \Lambda_g \rangle &= \langle g, p_\omega \psi(b) \rangle \\ &= \left\langle g, p_\omega \int_0^b (b-s)^{\alpha-1} K_\alpha(b-s) B u^*(s) ds \right\rangle \\ &= \int_0^b \langle g, p_\omega (b-s)^{\alpha-1} K_\alpha(b-s) B u^*(s) \rangle ds \\ &= \int_0^b \|B^*\varphi(t)\|^2 ds = \|g\|_G^2. \end{aligned}$$

Hence, $\Lambda: G \rightarrow E^\perp$ is one to one. It follows from Theorem 2.1 in [Lions, 1971] that (50) admits a unique solution in G .

Further, let $u = u^*$ in problem (1), one has $p_\omega z(b, u^*) = z_b$. Then for any $u_1 \in L^2(0, b, \mathbf{R}^p)$ with $p_\omega z(b, u_1) = z_b$, we obtain that $p_\omega [z(b, u^*) - z(b, u_1)] = 0$. Moreover, for any $g \in G$, we have $\langle g, p_\omega [z(b, u^*) - z(b, u_1)] \rangle = 0$ and

$$\begin{aligned} 0 &= \left\langle p_\omega^*g, \int_0^b (b-s)^{\alpha-1} K_\alpha(b-s) B (u^*(s) - u_1(s)) ds \right\rangle \\ &= \int_0^b \langle B^*(b-s)^{\alpha-1} K_\alpha^*(b-s) p_\omega^*g, u^*(s) - u_1(s) \rangle ds \\ &= \int_0^b \langle B^*\varphi(t), u^*(s) - u_1(s) \rangle ds. \end{aligned}$$

By the Theorem 1.3 in [Lions, 1971], it then follows from

$$\begin{aligned} J'(u^*) \cdot (u^* - u_1) &= 2 \int_0^b \langle u^*(s), u^*(s) - u_1(s) \rangle ds \\ &= 2 \int_0^b \langle B^*\varphi(t), u^*(s) - u_1(s) \rangle ds \\ &= 0, \end{aligned} \quad (52)$$

that u^* solves the minimum energy problem (37) and the proof is complete.

5 Examples

This section aims to present two examples to show the effectiveness of our obtained results.

Example 5.1.

Let us consider the following one dimensional time fractional order differential equations of order $\alpha \in (0, 1)$ with a zone actuator to show (1) of Remark 7.

$$\begin{cases} {}_0D_t^\alpha z(x, t) = \frac{\partial^2}{\partial x^2} z(x, t) + p_{[a_1, a_2]} u(t) \text{ in } [0, 1] \times [0, b], \\ \lim_{t \rightarrow 0^+} z(x, t) = z_0(x) \text{ in } [0, 1], \\ z(0, t) = z(1, t) = 0 \text{ in } [0, b], \end{cases} \quad (53)$$

where $Bu = p_{[a_1, a_2]}u$ and $0 \leq a_1 \leq a_2 \leq 1$. Moreover, we see that $-A = -\frac{\partial^2}{\partial x^2}$ with $\lambda_i = i^2\pi^2$, $\xi_i(x) = \sqrt{2} \sin(i\pi x)$, $\Phi(t)z = \sum_{i=1}^{\infty} \exp(-\lambda_i t)(z, \xi_i)_{L^2(0,1)} \xi_i$. and

$$K_\alpha(t)z(x) = \sum_{i=1}^{\infty} E_{\alpha, \alpha}(-\lambda_i t^\alpha)(z, \xi_i)_{L^2(0,1)} \xi_i(x).$$

Since $A = -\frac{\partial^2}{\partial x^2}$ is a self-adjoint operator, we have

$$\begin{aligned} (H^*z)(t) &= [B^*(b-t)^{\alpha-1} K_\alpha^*(b-t)z](t) \\ &= B^*(b-t)^{\alpha-1} \sum_{i=1}^{\infty} E_{\alpha, \alpha}(-\lambda_i(b-t)^\alpha)(z, \xi_i) \xi_i(x) \\ &= (b-t)^{\alpha-1} \sum_{i=1}^{\infty} E_{\alpha, \alpha}(-\lambda_i(b-t)^\alpha)(z, \xi_i) \int_{a_1}^{a_2} \xi_i(x) dx. \end{aligned}$$

By $\int_{a_1}^{a_2} \xi_i(x) dx = \frac{\sqrt{2}}{i\pi} \sin \frac{i\pi(a_1+a_2)}{2} \sin \frac{i\pi(a_2-a_1)}{2}$, we get that $\text{Ker}(H^*) \neq \{0\}$ ($\overline{\text{Im}(H)} \neq L^2(\omega)$) when $a_2 - a_1 \in \mathbf{Q}$. Then the system (53) is not controllable on $[0, 1]$.

Next, we show that there exists a sub-region $\omega \subseteq \Omega$ such that the system (53) is possible regional controllability in ω at time b .

Without loss of generality, let $a_1 = 0$, $a_2 = 1/2$, $z_* = \xi_k$, ($k = 4j, j = 1, 2, 3, \dots$). Based on the argument above, z_* is not reachable on $\Omega = [0, 1]$. However, since

$$E_{\alpha, \alpha}(t) > 0 \quad (t \geq 0) \quad \text{and} \quad \int_0^{1/2} \xi_i(x) dx = \frac{\sqrt{2}}{i\pi} (1 - \cos(i\pi/2)),$$

$i = 1, 2, \dots$, let $\omega = [1/4, 3/4]$, we see that

$$\begin{aligned} &(H^* p_\omega^* p_\omega z_*)(t) \\ &= \sum_{i=1}^{\infty} \frac{E_{\alpha, \alpha}(-\lambda_i(b-t)^\alpha)}{(b-t)^{1-\alpha}} (\xi_i, \xi_k)_{L^2(1/4, 3/4)} \int_0^{1/2} \xi_i(x) dx \\ &= \sum_{i \neq 4j} \frac{\sqrt{2} E_{\alpha, \alpha}(-\lambda_i(b-t)^\alpha)}{i\pi(b-t)^{1-\alpha}} \int_{1/4}^{3/4} \xi_i(x) \xi_{4j}(x) dx [1 - \cos(i\pi/2)] \\ &\neq 0. \end{aligned}$$

Then z_* is possible regional controllability in $\omega = [1/4, 3/4]$ at time b .

Example 5.2.

Consider the following time fractional differential equations with a pointwise actuator

$$\begin{cases} {}_0D_t^\alpha z(x, t) = \frac{\partial^2}{\partial x^2} z(x, t) + u(t) \delta(x - \sigma) \text{ in } [0, 1] \times [0, b], \\ \lim_{t \rightarrow 0^+} z(x, t) = z_0 \text{ in } [0, 1], \\ z(0, t) = z(1, t) = 0 \text{ in } [0, b], \end{cases} \quad (54)$$

which is excited by a pointwise control located at $\sigma \in [0, 1]$. Here $A = \frac{\partial^2}{\partial x^2}$ generates a strongly continuous semigroup. In addition, for any $g \in G$, by Lemma 11, we see that

$$g \rightarrow \|g\|_G^2 = \int_0^b \|\varphi(s)\|^2 ds$$

defines a norm on G , where φ is the unique mild solution of the following problem

$$\begin{cases} Q_t D_b^\alpha \varphi(t) = A^* Q \varphi(t), \quad t \in [0, b], \\ \lim_{t \rightarrow 0^+} Q_t D_b^{\alpha-1} \varphi(t) = p_\omega^* g. \end{cases} \quad (55)$$

Now if we consider the following system

$$\begin{cases} {}_0D_t^\alpha \psi(t) = A\psi(t) + \delta(x - \sigma)\varphi(\sigma, t), \quad t \in [0, b], \\ \lim_{t \rightarrow 0^+} {}_0D_t^{\alpha-1} \psi(t) = 0. \end{cases} \quad (56)$$

Let $\omega \subseteq [0, 1]$ be a subinterval and let $\Lambda: G \rightarrow H^1$ be

$$\Lambda_g = p_\omega \psi(b). \quad (57)$$

Then the regional controllability of the example (54) is equivalent to solving the equation

$$\Lambda_g := z_b - p_\omega \tilde{\psi}(b), \quad \forall z_b \in L^2(\omega) \quad (58)$$

where $\tilde{\psi}(t)$ is the solution of the following system

$$\begin{cases} {}_0D_t^\alpha \tilde{\psi}(t) = A\tilde{\psi}(t), & t \in [0, b], \\ \lim_{t \rightarrow 0^+} \tilde{\psi}(t) = z_0. \end{cases} \quad (59)$$

Thus, by Theorem 12, we can conclude that if the example (54) is regionally approximately controllable on some subregion of $[0, 1]$, for any $z_b \in L^2(\omega)$, (58) admits a unique solution $g \in G$. Moreover, the control

$$\begin{aligned} u^*(t) &= \varphi(\sigma, t) \\ &= \sum_{i=1}^{\infty} (b-t)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i(b-t)^\alpha) (p_\omega^* g, \xi_i) \xi_i(\sigma) \end{aligned}$$

steers (54) to z_b at time b and u^* solves the minimum control energy problem (37).

6 CONCLUSIONS

The purpose of this paper is to investigate the regional controllability of the Riemann-Liouville time fractional diffusion equations of order $\alpha \in (0, 1)$. The characterizations of strategic actuators when the control inputs appear in the differential equations as distributed inputs and an approach on the regional controllability with minimum energy of the problems (1) are solved. Since $E_1(t) = e^t$, $t \geq 0$, together with (27), we get that our results can be regarded as the extension of the results in [El Jai et al., 1995] and [Sakawa, 1974].

Moreover, the results presented here can also be extended to complex fractional order distributed parameter dynamic systems. For instance, the problem of constrained regional control of fractional order diffusion systems with more complicated regional sensing and actuation configurations are of great interest. For more information on the potential topics related to fractional distributed parameter systems, we refer the readers to [Ge et al., 2015] and the references therein.

Acknowledgements

This work was supported by the Natural Science Foundation of Shanghai (No.15ZR1400800).

References

- [Chen & Feng, 2016] Chen, X., & Feng Y. (2016). Adaptive control for continuous-time systems with actuator and sensor hysteresis. *Automatica* 64, 196–207.
- [Courant & Hilbert, 1966] Courant, R., & Hilbert, D. (1966). *Methods of mathematical physics* volume 1. CUP Archive.
- [Dacorogna, 2007] Dacorogna, B. (2007). *Direct methods in the calculus of variations, Second edition* volume 78. Springer Science & Business Media.
- [El Jai, 1991] El Jai, A. (1991). Distributed systems analysis via sensors and actuators. *Sensors and Actuators A: Physical*, 29, 1–11.
- [El Jai & Pritchard, 1988] El Jai, A., & Pritchard, A. J. (1988). *Sensors and controls in the analysis of distributed systems*. Halsted Press.
- [El Jai et al., 1995] El Jai, A., Simon, M., Zerrik, E., & Pritchard, A. (1995). Regional controllability of distributed parameter systems. *International Journal of Control*, 62, 1351–1365.
- [Erdélyi et al., 1953] Erdélyi, A., Magnus, W., Oberhettinger, F., & Tricomi, F. G. (1953). Higher transcendental functions, Vol.1. *McGraw-Hill Book Company, 1953*.
- [Fujishiro & Yamamoto, 2014] Fujishiro, K., & Yamamoto, M. (2014). Approximate controllability for fractional diffusion equations by interior control. *Applicable Analysis*, 93, 1793–1810.
- [Ge et al., 2015] Ge, F., Chen, Y., & Kou, C. (2015). Cyber-physical systems as general distributed parameter systems: three types of fractional order models and emerging research opportunities. *IEEE/CAA Journal of Automatica Sinica*, 2, 353–357.
- [Ge et al., 2016a] Ge, F., Chen, Y., & Kou, C. (2016a). Actuator characterizations to achieve approximate controllability for a class of fractional sub-diffusion equations. *International Journal of Control*, 1–9.
- [Ge et al., 2016b] Ge, F., Chen, Y., & Kou, C. (2016b). Regional gradient controllability of sub-diffusion processes. *Journal of Mathematical Analysis and Applications*, 440, 865–884.
- [Gorenflo et al., 2014] Gorenflo, R., Kilbas, A. A., Mainardi, F., & Rogosin, S. V. (2014). *Mittag-Leffler functions, related topics and applications*. Springer.
- [Hilfer, 2000] Hilfer, R. (2000). *Applications of fractional calculus in physics* volume 128. World Scientific.
- [Hilfer & Anton, 1995] Hilfer, R., & Anton, L. (1995). Fractional master equations and fractal time random walks. *Physical Review E*, 51, 848–851.
- [Kilbas et al., 2006] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations*. Elsevier Science Limited.
- [Lin & Lu, 2013] Lin, S.-D., & Lu, C.-H. (2013). Laplace transform for solving some families of fractional differential equations and its applications. *Advances in Difference Equations*, 2013, 1–9.
- [Lions, 1971] Lions, J. L. (1971). *Optimal control of systems governed by partial differential equations* volume 170. Springer Verlag.
- [Mainardi et al., 2007] Mainardi, F., Mura, A., Pagnini, G., & Gorenflo, R. (2007). Sub-diffusion equations of fractional order and their fundamental solutions. In *Mathematical methods in engineering* (pp. 23–55). Springer.
- [Mainardi et al., 2007] Mainardi, F., Paradisi, P., & Gorenflo, R. (2007). Probability distributions generated by fractional diffusion equations. 2007. *arXiv preprint arXiv:0704.0320*.
- [Klimek, 2009] Klimek, M. (2009). *On solutions of linear fractional differential equations of a variational type*. Publishing Office of Czestochowa University of Technology.
- [Mandelbrot, 1983] Mandelbrot, B. B. (1983). *The fractal geometry of nature* volume 173. Macmillan.
- [Mathai & Haubold, 2008] Mathai, A. M., & Haubold, H. J. (2008). *Special functions for applied scientists* volume 4. Springer.

- [Metzler & Klafter, 2000] Metzler, R., & Klafter, J. (2000). The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Physics Reports*, 339, 1–77.
- [Montroll & Weiss, 1965] Montroll, E. W., & Weiss, G. H. (1965). Random walks on lattices. II. *Journal of Mathematical Physics*, 6, 167–181.
- [Pritchard & Wirth, 1978] Pritchard, A., & Wirth, A. (1978). Unbounded control and observation systems and their duality. *SIAM Journal on Control and Optimizatio*, 16, 535–545.
- [Renardy & Rogers, 2006] Renardy, M., & Rogers, R. C. (2006). *An introduction to partial differential equations* volume 13. Springer Science & Business Media.
- [Sakawa, 1974] Sakawa, Y. (1974). Controllability for partial differential equations of parabolic type. *SIAM Journal on Control and Optimizatio*, 12, 389–400.
- [Spears & Spears, 2012] Spears, W. M., & Spears, D. F. (2012). *Physicomimetics: Physics-based swarm intelligence*. Springer Science & Business Media.
- [Uchaikin & Sibatov, 2012] Uchaikin, V., & Sibatov, R. (2012). Fractional kinetics in solids: Anomalous charge transport in semiconductors. *Dielectrics and Nanosystems (World Science, 2013)*, .
- [Weinberger, 1962] Weinberger, H. (1962). Symmetrization in uniformly elliptic problems. *Studies in Mathematical Analysis, Stanford University. Press*, (pp. 424–428).