Taylor & Francis
Taylor & Francis Group

International Journal of Control

ISSN: 0020-7179 (Print) 1366-5820 (Online) Journal homepage: http://www.tandfonline.com/loi/tcon20

Actuator characterisations to achieve approximate controllability for a class of fractional sub-diffusion equations

Fudong Ge, YangQuan Chen & Chunhai Kou

To cite this article: Fudong Ge, YangQuan Chen & Chunhai Kou (2016): Actuator characterisations to achieve approximate controllability for a class of fractional sub-diffusion equations, International Journal of Control, DOI: 10.1080/00207179.2016.1163619

To link to this article: http://dx.doi.org/10.1080/00207179.2016.1163619

Accepted author version posted online: 10 Mar 2016.
Published online: 13 Apr 2016.

INTERNATIONAL JOURNAL OF CONTROL, 2016
http://dx.doi.org/10.1080/00207179.2016.1163619

Taylor & Francis
Taylor & Francis Group

Actuator characterisations to achieve approximate controllability for a class of fractional sub-diffusion equations

Fudong Ge[a], YangQuan Chen[b] and Chunhai Kou[c]

[a]College of Information Science and Technology, Donghua University, Shanghai, P.R. China; [b]Mechatronics, Embedded Systems and Automation Lab, University of California, Merced, CA, USA; [c]Department of Applied Mathematics, Donghua University, Shanghai, P.R. China

ABSTRACT

This paper is devoted to analysing the actuator characterisations for the fractional sub-diffusion equation under consideration to become approximately controllable. Two different cases are considered, where the control inputs emerge in the differential equation as distributed inputs and as boundary inputs in the boundary conditions. The dual system for fractional sub-diffusion equation is solved and the necessary and sufficient conditions for the approximate controllability of the system are established. Several examples are worked out in the end to illustrate our results.

ARTICLE HISTORY
Received 9 May 2015
Accepted 5 March 2016

KEYWORDS
Approximate controllability; sub-diffusion process; fractional order; distributed inputs; boundary inputs

1. Introduction

Let Ω be an open bounded subset of $\mathbf{R}^n$ with smooth boundary $\Sigma = \partial\Omega$ and we consider the following abstract fractional sub-diffusion equations excited by p actuators:

$$\begin{cases} {}_0^C D_t^\alpha z(t,x) + Az(t,x) & \text{in} \quad [0,b]\times\Omega, \\ \quad = \sum_{i=1}^{p} g_i(x)u_i(t) & \\ z(t,\eta) = 0 & \text{on} \quad [0,b]\times\Sigma, \\ z(0,x) = z_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $z_0 \in L^2(\Omega)$, $b > 0$ is an arbitrary fixed value and $z = z(t, x)$ is the state to be controlled. In addition, $p \in \mathbf{N}$ is the number of actuators, $g_i(x) \in L^2(\Omega)$ are the spatial distribution of actuators and $u_i(t) \in L^2(0, b)$ are the control inputs generated by actuators, $i = 1, 2, \dots, p$. It is supposed that A is a symmetric and uniformly elliptic operator and we refer the reader to Renardy and Rogers (2006) and Sario and Weill (1965) for more properties on operator A. Here, ${}_0^C D_t^\alpha$ and ${}_0I_t^\alpha$ denote the Caputo fractional derivative and the Riemann–Liouville fractional integral with respect to time t, respectively, defined by Podlubny (1999) and Kilbas, Srivastava, and Trujillo (2006)

$${}_0^C D_t^\alpha z(t,x) = \begin{cases} {}_0I_t^{n-\alpha} \frac{\partial^n}{\partial t^n} z(t,x), & n-1<\alpha<n, \\ \frac{\partial^n}{\partial t^n} z(t,x), & \alpha = n,\ n\in\mathbf{N} \end{cases} \tag{1.2}$$

and

$${}_0I_t^\alpha z(t,x) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} z(s,x)ds,\ \alpha>0. \tag{1.3}$$

In order to state the results, since A is a symmetric and uniformly elliptic operator, by Courant and Hilbert (1966), we get that there exists a sequence (λ_j, ξ_{jk}): $k = 1, 2, \dots, r_j, j = 1, 2, \dots$, such that

(1) For each $j = 1, 2, \dots, \lambda_j$ is the eigenvalue of operator A with multiplicities r_j and

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots, \quad \lim_{j\to\infty} \lambda_j = \infty.$$

(2) For each $j = 1, 2, \dots, \xi_{jk} (k = 1, 2, \dots, r_j)$ are the orthonormal eigenfunctions corresponding to λ_j, i.e.,

$$(\xi_{jk_m}, \xi_{jk_n}) = \begin{cases} 1, & k_m = k_n, \\ 0, & k_m \neq k_n, \end{cases}$$

where $1 \le k_m, k_n \le r_j$, $k_m, k_n \in \mathbf{N}$ and $(\cdot,\cdot)$ is the inner product of space $L^2(\Omega)$.

It is well known that the sequence $\{\xi_{jk}, k = 1, 2, \dots, r_j, j = 1, 2, \dots\}$ is an' orthonormal basis in $L^2(\Omega)$ and for any $z_1(x) \in L^2(\Omega)$, it can be expressed as

$$z_1(x) = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j} (z_1, \xi_{jk})\xi_{jk}(x).$$

CONTACT YangQuan Chen ychen53@ucmerced.edu

Recently, the fractional anomalous sub-diffusion equations, which replace the traditional first-order time derivative by a fractional derivative of order $\alpha \in (0, 1)$, have been proved to be a useful tool in modelling those sub-diffusion process (Cao, Li, & Chen, 2015; Metzler & Klafter, 2000, 2004), since the introduction of continuous time random walks (CTRWs) by Montroll and Weiss (1965). More precisely, the CTRWs, governed by the waiting time probability density function (PDF) and the jump length PDF, is popular in statistical physics to model anomalous diffusion, where a cloud of particles spreads at a different manner compared with the traditional diffusion equation predicts (Adams & Gelhar, 1992; Cartea & Negrete, 2007; Meerschaert & Scalas, 2006). Moreover, when the particles are assumed to jump at fixed time intervals with incorporating waiting times, the particles will experience the sub-diffusion process and the fractional sub-diffusion equation can be introduced to well characterise those sub-diffusion process such as the flow through porous media microscopic processes (Uchaikin & Sibatov, 2013), or the swarm of robots moving through dense forest (Spears & Spears, 2012), etc.

As for the notions of sensors and actuators, it was El Jai and Pritchard who introduced them in the 1980s to better describe the measurements and actions (El Jai & Pritchard, 1988), which mainly focus on the locations, number and spatial distributions of sensors and actuators. More precisely, as cited in Coopmans, Stark, Jensen, Chen, and McKee (2015), all these distributed parameter systems with moving sensors and actuators form the so-called cyber-physical systems, which are rich in real-world applications. The motivation of this study is from Cao, Chen, and Li (2014) where the pest spreading process was investigated with mobile sensors and actuators and from Cao, Chen, Stuart, and Yue (2016) where the authors considered the egress or evacuation of crowd-pedestrian by using the notions of sensors and actuators. Now it is now widely believed that using fractional calculus in modelling can better capture the complex dynamics of natural and man-made systems, and fractional order controls can offer better performance not achievable before using integer order controls (Mandelbrot, 1983; Monje, Chen, Vinagre, Xue, & Liu, 2010; Torvik & Bagley, 1984). Furthermore, for more knowledge on fractional calculus and fractional order partial differential equations, we refer the reader to Podlubny (1999), Kilbas et al. (2006), Agrawal (2002), Kilbas and Trujillo (2002), and Sakamoto and Yamamoto (2011) and the references therein.

The contribution of this paper is on the actuator characterisations when the anomalous sub-diffusion process described by fractional sub-diffusion equation is approximately controllable. To this purpose, we first obtain the expression of the unique mild solution of the system (1.1). Then the necessary and sufficient conditions for the approximate controllability of (1.1) are established when the control inputs appear in the differential equations as distributed inputs. In addition, we solve the dual system for the fractional diffusion equation and obtain the necessary and sufficient conditions for the controllability of the fractional sub-diffusion equation with inputs emerging in the boundary conditions. Moreover, it should be pointed out that the latter form of control system is easier to be realised physically.

The rest of the paper is organised as follows. Some preliminaries results are presented in the next section. In Sections 3 and 4, our main results are shown and proved. Finally, several illustrations are worked out.

2. Preliminary results

In this section, we shall introduce the following definition and some lemmas to be used thereafter.

Definition 2.1 (Fujishiro & Yamamoto, 2014; Sakamoto & Yamamoto, 2011)**:** *For any given $u = (u_1, u_2, \dots, u_p)$, $u_i(t) \in L^2(0, b)$, a function $z \in L^2([0, b] \times \Omega)$ is said to be a mild solution of (1.1) if it satisfies*

$$z(t,x) = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\Bigg[E_\alpha(-\lambda_j t^\alpha) z^0_{jk} + \sum_{i=1}^{p}\int_0^t \frac{E_{\alpha,\alpha}(-\lambda_j (t-\tau)^\alpha)}{(t-\tau)^{1-\alpha}} u_i(\tau)\, d\tau\, g^i_{jk}\Bigg]\xi_{jk}(x), \quad (2.1)$$

where $z^0_{jk} = (z_0, \xi_{jk})$, $g^i_{jk} = (g_i, \xi_{jk})$, $j = 1, 2, \dots, k = 1, 2, \dots, r_j$ and

$$E_\alpha(z) := \sum_{i=0}^{\infty}\frac{z^i}{\Gamma(\alpha i + 1)}, \quad E_{\alpha,\beta}(z) := \sum_{i=0}^{\infty}\frac{z^i}{\Gamma(\alpha i + \beta)}, \quad \mathbf{R}(\alpha) > 0,\ \beta, z \in \mathbf{C}$$

are known as the Mittag–Leffler function and the generalised Mittag–Leffler function in two parameters, respectively.

Lemma 2.1: *For any $\alpha > 0$, the usual n −derivatives of $E_\alpha(-\lambda z^\alpha)$ are*

$$\left(\frac{d}{dz}\right)^n [E_\alpha(-\lambda z^\alpha)] = -\lambda z^{\alpha-n} E_{\alpha,\alpha-n+1}(-\lambda z^\alpha), \quad \lambda > 0,\ n \in \mathbf{N}. \quad (2.2)$$

Moreover, we have

$${}_0I_z^{1-\alpha}\left(z^{\alpha-1}E_{\alpha,\alpha}(-\lambda z^\alpha)\right) = E_\alpha(-\lambda z^\alpha),\ \lambda > 0,\ \alpha \in (0,1). \quad (2.3)$$

Proof: For any $\alpha > 0$, since $E_{\alpha,\beta}(z)$ is an entire function of z, one has

$$\begin{aligned}\left(\frac{d}{dz}\right)^n [E_\alpha(-\lambda z^\alpha)] &= \left(\frac{d}{dz}\right)^n \left[1 + \sum_{k=1}^{\infty} \frac{(-\lambda z^\alpha)^k}{\Gamma(\alpha k + 1)}\right] \\ &= \sum_{k=1}^{\infty} \frac{(-\lambda)^k z^{\alpha k - n}}{\Gamma(\alpha k - n + 1)} \\ &= -\lambda z^{\alpha - n} \sum_{k=1}^{\infty} \frac{(-\lambda)^{k-1} z^{\alpha(k-1)}}{\Gamma(\alpha(k-1) + \alpha - n + 1)} \\ &= -\lambda z^{\alpha-n} E_{\alpha,\alpha-n+1}(-\lambda z^\alpha)\end{aligned}$$

and when $\alpha \in (0, 1)$, we see

$$\begin{aligned}{}_0I_z^{1-\alpha}\left(z^{\alpha-1}E_{\alpha,\alpha}(-\lambda z^\alpha)\right) &= \frac{1}{\Gamma(1-\alpha)} \\ &\quad\times \int_0^z (z-s)^{-\alpha} s^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-\lambda)^k s^{\alpha k}}{\Gamma(\alpha k + \alpha)} ds \\ &= \sum_{k=0}^{\infty} \int_0^z \frac{(-\lambda)^k (z-s)^{-\alpha} s^{\alpha k + \alpha - 1}}{\Gamma(1-\alpha)\Gamma(\alpha k + \alpha)} ds \\ &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k z^{\alpha k}}{\Gamma(\alpha k + 1)} \\ &= E_\alpha(-\lambda z^\alpha).\end{aligned}$$

The proof is complete. □

Lemma 2.2: *Let $\varphi, \psi : \mathbf{R}^+ \to \mathbf{R}$ and suppose that ${}_0I_t^\alpha \varphi$ and ${}_0I_t^\alpha \psi$ $(\alpha > 0)$ exist, then we have*

$$\varphi * \left({}_0I_t^\alpha \psi\right) = \psi * \left({}_0I_t^\alpha \varphi\right), \tag{2.4}$$

where $\varphi*\psi$ stands for the convolution of two functions, such that

$$\varphi * \psi = \int_0^t \varphi(t-s)\psi(s)ds. \tag{2.5}$$

Proof: For any $t > 0, \alpha > 0$, by direct computing, one has

$$\begin{aligned}\varphi * \left({}_0I_t^\alpha \psi\right) &= \int_0^t \varphi(t-s) \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \psi(\tau) d\tau ds \\ &= \int_0^t \int_\tau^t \varphi(t-s) \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} ds \psi(\tau) d\tau \\ &= \int_0^t \psi(\tau) \int_0^{t-\tau} \frac{(t-\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\sigma) d\sigma d\tau \\ &= \psi * \left({}_0I_t^\alpha \varphi\right).\end{aligned}$$

This completes the proof. □

Lemma 2.3 (Dacorogna, 2008)**:** *Let $\Omega \subseteq \mathbf{R}^n$ be an open set and $C_0^\infty(\Omega)$ be the class of infinitely differentiable functions on Ω with compact support in Ω and $u \in L^1_{loc}(\Omega)$ be such that*

$$\int_\Omega u(x)\psi(x)dx = 0, \quad \forall \psi \in C_0^\infty(\Omega). \tag{2.6}$$

Then $u = 0$ almost everywhere in Ω.

3. Controllability analysis by interior inputs

The system (1.1) is said to be approximately controllable at arbitrary given time $b > 0$ if for any $z_0 \in L^2(\Omega)$, $v(x) \in L^2(\Omega)$, given $\varepsilon > 0$, there exists a control $u := (u_1, u_2, \ldots, u_p) \in L^2(0, b; \mathbf{R}^p)$ $(u_i \in L^2(0, b), i = 1, 2, \ldots, p)$, such that

$$\|z(b, \cdot) - v\|_{L^2(\Omega)} \le \varepsilon \tag{3.1}$$

for some b depending in general on v and ε.

For $u \in L^2(0, b; \mathbf{R}^p)$, $t \ge 0$, consider the following attainable set $D(t)$ in $L^2(\Omega)$:

$$D(t) = \left\{ d(t,\cdot) \in L^2(\Omega) : d(t,x) = \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} \sum_{i=1}^{p} \times \int_0^t \frac{E_{\alpha,\alpha}(-\lambda_j(t-\tau)^\alpha)}{(t-\tau)^{1-\alpha}} u_i(\tau) d\tau g_{jk}^i \xi_{jk}(x) \right\}$$

and we obtain the following proposition.

Proposition 3.1: *For any given $b > 0$, the necessary and sufficient condition for the approximate controllability of the system (1.1) at time b is that*

$$\overline{D(b)} = L^2(\Omega),$$

where $D(t)$ $(t > 0)$ is a linear manifold and $\overline{D(b)}$ is the closure of $D(b)$.

Proof: Similar to the argument in Fujishiro and Yamamoto (2014), for any given $b > 0$, if $u_i \equiv 0$ for all $i = 1, 2, \ldots, p$ in system (1.1), by Sakamoto and Yamamoto (2011), there exists a unique solution $\omega(z_0) \in C([0, b]; L^2(\Omega)) \cap C((0, b]; H^2(\Omega) \cap H_0^1(\Omega))$ to (1.1), such that

$$\omega(z_0)(t, x) = \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} E_\alpha(-\lambda_j t^\alpha)(z_0, \xi_{jk})\xi_{jk}(x), \tag{3.2}$$

where the series (3.2) is convergent. Moreover, there exists a constant $c > 0$ satisfying $\|\omega(z_0)\|_{C([0,b];L^2(\Omega))} \le c\|z_0\|_{L^2(\Omega)}$. Then we see that (3.2) is well defined.

For any $v \in L^2(\Omega)$, since $\omega(z_0)(b, \cdot) \in L^2(\Omega)$, we have $(v - \omega(z_0))(b, \cdot) \in L^2(\Omega)$. If $\overline{D(b)} = L^2(\Omega)$, for any $\varepsilon > 0$, we can find $u = (u_1, u_2, \ldots, u_p) \in L^2(0, b; \mathbf{R}^p)$ satisfying

$$\|d(b, \cdot) - (v - \omega(z_0))\,(b, \cdot)\|_{L^2(\Omega)} < \varepsilon.$$

This implies $\|z(b, \cdot) - v\|_{L^2(\Omega)} < \varepsilon$, where $z(t, x) = d(t, x) + \omega(z_0)(t, x)$ solves the system (1.1) with the control inputs u. Then (1.1) is approximately controllable at time b.

On the contrary, for arbitrarily given $b > 0$, by Theorem 2.1 in Fujishiro and Yamamoto (2014), we get that the system (1.1) is approximately controllable at time b if and only if

$$\overline{\{z(b, x) : u \in L^2\,(0, b; \mathbf{R}^p)\}} = L^2(\Omega),$$

i.e., for any $v \in L^2(\Omega)$, given $\varepsilon > 0$, there exists a control $u \in L^2(0, b; \mathbf{R}^p)$, such that

$$\|z(b, \cdot) - v\|_{L^2(\Omega)} = \|(z - \omega(z_0))(b, \cdot) - (v - \omega(z_0))(b, \cdot)\|_{L^2(\Omega)} \leq \varepsilon. \tag{3.3}$$

By Definition 2.1, we obtain that $(z - \omega(z_0))(b, \cdot) \in D(b)$. Then $(v - \omega(z_0))(b, \cdot) \in L^2(\Omega)$ gives $\overline{D(b)} = L^2(\Omega)$. The proof is complete. □

Thus, by Proposition 3.1, it suffices to suppose that $z_0 \equiv 0$ in the following discussion. Moreover, for any $z_* \in \left[\overline{D(b)}\right]^{\perp}$, $d(b, \cdot) \in D(b)$, we have

$$(d(b, \cdot), z_*) = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{p}\int_0^b (b - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j(b - \tau)^{\alpha})u_i(\tau)d\tau g^i_{jk} z_{jk} = 0, \tag{3.4}$$

where $z_{jk} = (\xi_{jk}, z_*)$, $j = 1, 2, \ldots$, $k = 1, 2, \ldots, r_j$, by Lemma 2.3, since $u = (u_1, u_2, \ldots, u_p)$ in (3.4) is arbitrary and $t^{\alpha-1}E_{\alpha}(-\lambda_j t^{\alpha}) > 0$ for all $t \geq 0$, one has

$$\sum_{j=1}^{\infty}\sum_{k=1}^{r_j} t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j t^{\alpha}) g^i_{jk} z_{jk} = 0 \text{ for all } t \in [0, b], \quad i = 1, 2, \ldots, p. \tag{3.5}$$

Then we conclude that the necessary and sufficient condition for approximate controllability of the system (1.1) at time b is that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{r_j} t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j t^{\alpha}) g^i_{jk} z_{jk} = 0 \text{ for all } t \in [0, b], \quad i = 1, 2, \ldots, p \text{ implies } z_* = 0. \tag{3.6}$$

Theorem 3.1: *For any $j = 1, 2, \ldots$, arbitrarily given $b > 0$, define $p \times r_j$ matrices G_j as*

$$G_j = \begin{bmatrix} g^1_{j1} & g^1_{j2} & \cdots & g^1_{jr_j} \\ g^2_{j1} & g^2_{j2} & \cdots & g^2_{jr_j} \\ \vdots & \vdots & \vdots & \vdots \\ g^p_{j1} & g^p_{j2} & \cdots & g^p_{jr_j} \end{bmatrix}, \tag{3.7}$$

where $g^i_{jk}(k = 1, 2, \ldots, r_j, i = 1, 2, \ldots, p)$ are defined in Definition 2.1. Then the system (1.1) is approximately controllable at time b if and only if

$$p \geq r = \max\{r_j\} \quad \text{and} \quad \operatorname{rank} G_j = r_j \text{ for any } j = 1, 2, \ldots. \tag{3.8}$$

Proof: With the above arguments, given $b > 0$, it follows that the system (1.1) is approximately controllable at time b if and only if

$$\sum_{j=1}^{\infty}\sum_{k=1}^{r_j} t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j t^{\alpha}) g^i_{jk} z_{jk} = 0 \text{ for } t \in [0, b], \quad i = 1, 2, \ldots, p \Rightarrow z_* = 0,$$

i.e., for any $z_* \in \left[\overline{D(b)}\right]^{\perp}$, one has

$$\sum_{j=1}^{\infty} t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j t^{\alpha}) G_j z_j = \theta,\ t \in [0, b], \quad i = 1, 2, \ldots, p \ \Rightarrow z_* = 0, \tag{3.9}$$

where $\theta = (0, 0, \ldots, 0) \in \mathbf{R}^p$, $z_j = (z_{j1}, z_{j2}, \ldots, z_{jr_j})^T$ is a vector in $\mathbf{R}^{r_j}$ and $j = 1, 2, \ldots$.

(1) If the system (1.1) is not regionally approximately controllable at time b, i.e., $\overline{D(b)} \neq L^2(\Omega)$. There exists a nonzero element $z* \in L^2(\Omega)$, such that

$$\left(z(b, \cdot), z^*\right) = 0 \text{ for all } u \in L^2(0, b; \mathbf{R}^p). \tag{3.10}$$

Then we can find a nonzero element $z_{j^*k} \neq 0$ satisfying

$$G_{j^*} z_{j^*} = \theta. \tag{3.11}$$

Thus, if $p \geq r = \max\{r_j\}$, we see that

$$\operatorname{rank} G_{j^*} < r_{j^*}. \tag{3.12}$$

(2) On the contrary, if $p \geq r = \max\{r_j\}$ and $\operatorname{rank} G_j < r_j$ forsome $j = 1, 2, \ldots$, there exists a nonzero element $\tilde{z} \in L^2(\omega)$ with $\tilde{z}_j = \left(\tilde{z}_{j1}, \tilde{z}_{j2}, \ldots, \tilde{z}_{jr_j}\right)^T \in \mathbf{R}^{r_j}$, such that

$$G_j \tilde{z}_j = \theta. \tag{3.13}$$

Then there exists a nonzero element $\tilde{z} \in D^{\perp}$ satisfying

$$\sum_{j=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^{\alpha}) G_j \tilde{z}_j = \theta. \tag{3.14}$$

This implies that $\bar{D} \neq L^2(\Omega)$ and the system (1.1) is not approximately controllable at time b. The proof is complete. □

Remark 3.1:

(1) The system (1.1) with $\alpha = 1$,

$$A = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) + q(x)$$

and $q(x)$ being Hölder continuous on the compact domain of $\mathbf{R}^n$ is considered in Sakawa (1974), which can be regarded as a particular case of Theorem 3.1.

(2) If the multiplicity of the eigenvalues λ_j of operator A is infinite for some $j = 1, 2, \ldots$ and if the system (1.1) is approximately controllable, then the number of the control functions should also be infinite.

4. Controllability analysis by boundary inputs

The purpose of this section is to investigate the following fractional sub-diffusion equations with the control inputs emerging in the boundary condition:

$$\begin{cases} {}_0^C D_t^{\alpha} y(t,x) + Ay(t,x) = 0 & \text{in } (0,b)\times\Omega, \\ y(t,\eta) = \sum_{i=1}^{q} h_i(x) u_i(t) & \text{on } (0,b)\times\Sigma, \\ y(0,x) = 0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

where $0 < \alpha < 1$, $q \in \mathbf{N}$, $h_i \in L^2(\Omega)$, $u_i \in L^2(0,b)$, $i = 1, 2, \ldots, q$ and A is a symmetric, uniformly elliptic operator.

4.1 Dual system

In this part, we first explore the dual relationship between the following system

$$\begin{cases} A\varphi = 0 & \text{in } \Omega, \\ \varphi = h & \text{on } \Sigma \end{cases} \tag{4.2}$$

and its dual system

$$\begin{cases} A^*\psi = f & \text{in } \Omega, \\ \psi = 0 & \text{on } \Sigma, \end{cases} \tag{4.3}$$

where h is given on Σ, f is given in Ω. It is well known that for any $h \in L^2(\Sigma)$ and $f \in L^2(\Omega)$, (4.2) and (4.3) possess unique solution φ, $\psi \in L^2(\Omega)$. Henceforth, we will denote the solution of (4.2) by φ_h and the solution of (4.3) by ψ_f, respectively. In addition, in a weak sense, from the argument in Appendix A of Fujishiro (2015), multiplying both sides of (4.3) with $\varphi(x)$ and integrating in $L^2(\Omega)$, by the Green's formula, we can get the following lemma.

Lemma 4.1 (Fujishiro, 2015): *For any $f \in L^2(\Omega)$, we have*

$$\int_{\Omega} \varphi_h(x) f(x) dx = \int_{\Sigma} h(x) \frac{\partial \psi_f}{\partial \varsigma}(x) d\sigma_x, \tag{4.4}$$

where $\varsigma(x) = (\varsigma_1(x), \ldots, \varsigma_n(x))$ is the outward unit normal vector to Σ at x. In addition, by replacing f with ξ_{jk}, we have

$$\lambda_j(\varphi_h(x), \xi_{jk}(x)) = \left\langle h(x), \frac{\partial}{\partial\varsigma}\xi_{jk}(x)\right\rangle, \tag{4.5}$$

where $j = 1, 2, \ldots$, $k = 1, 2, \ldots, r_j$ and $<\cdot,\cdot>$ is the inner products in space $L^2(\Sigma)$.

Next, we shall study the representation of the solution to the problem (4.1).

Lemma 4.2: *For any given $u = (u_1, u_2, \ldots, u_q)$, $u_i \in L^2(0,b)$, the unique mild solution of (4.1) can be expressed as*

$$y(t,x) = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{q} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j(t-\tau)^{\alpha}) \times u_i(\tau) d\tau \left\langle h_i, \frac{\partial}{\partial\varsigma}\xi_{jk}\right\rangle \xi_{jk}(x). \tag{4.6}$$

For the proof of Lemma 4.2, we refer to Fujishiro (2015), where the authors studied the approximate controllability of the time fractional order diffusion system with boundary control and obtained the unique existence of mild solutions of the problem under consideration. Here, we investigate the unique existence of the mild solution of the systems (4.1), which are not exactly the same but the proof is based on similar arguments. For the reader's convenience, the proofs of Lemma 4.2 will be given in the Appendix.

Similar to argument in the preceding section, consider the following attainable set $K(t)$ in $L^2(\Omega)$:

$$K(t) = \left\{ y(t,\cdot) \in L^2(\Omega) : y(t,x) = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{q} \times \int_0^t \frac{E_{\alpha,\alpha}(-\lambda_j(t-\tau)^{\alpha})}{(t-\tau)^{1-\alpha}} u_i(\tau) d\tau h_{jk}^i \xi_{jk}(x) \right\},$$

where $h^i_{jk} = \langle h_i, \frac{\partial}{\partial\varsigma}\xi_{jk}\rangle$. Moreover, for arbitrarily given $b > 0$, by Theorem 2.1 in Fujishiro (2015), the system (4.1) is approximately controllable at time b if and only if $\overline{\{y(b,x) : u \in L^2(0,b;\mathbf{R}^p)\}} = L^2(\Omega)$. Since $y(0,x) = 0$ and $\overline{\{y(b,x) : u \in L^2(0,b;\mathbf{R}^p)\}} \Leftrightarrow \overline{K(b)}$, similar to the proof of Proposition 3.1, we then have the following proposition.

Proposition 4.1: *For any given $b > 0$, the system (4.1) is approximately controllable at time b if and only if*

$$\overline{K(b)} = L^2(\Omega),$$

where $K(t)$ $(t > 0)$ is a linear manifold and $\overline{K(b)}$ is the closure of $K(b)$.

Furthermore, for any $y^* \in \left[\overline{K(b)}\right]^\perp$, we have

$$(y(b,\cdot), y^*) = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{q} \times \int_0^b \frac{E_{\alpha,\alpha}(-\lambda_j(b-\tau)^\alpha)}{(b-\tau)^{1-\alpha}} u_i(\tau)d\tau h^i_{jk} y^*_{jk} = 0 \quad (4.7)$$

for all $y(b,\cdot) \in K(b)$, where $y^*_{jk} = (\xi_{jk}, y^*)$. Then Lemma 2.3, Proposition 4.1, together with Equation (4.7) show that the system (4.1) with the control inputs appearing in the boundary condition is completely approximately controllable at time $b \Leftrightarrow$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{r_j} t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j t^\alpha) h^i_{jk} y_{jk} = 0 \text{ for all } t \in [0,b], \quad i = 1,2,\cdots,q. \text{ implies } y^* = 0.$$

Thus, we get the following theorem. Because the proofs are similar to the ones in Theorem 3.1, we omit the details.

Theorem 4.1: *For any $j = 1, 2, \dots$, given arbitrary $b > 0$, define $q \times r_j$ matrices H_j as*

$$H_j = \begin{bmatrix} h^1_{j1} & h^1_{j2} & \cdots & h^1_{jr_j} \\ h^2_{j1} & h^2_{j2} & \cdots & h^2_{jr_j} \\ \vdots & \vdots & \vdots & \vdots \\ h^q_{j1} & h^q_{j2} & \cdots & h^q_{jr_j} \end{bmatrix}, \quad (4.8)$$

where $h^i_{jk} = \langle h_i, \frac{\partial}{\partial\varsigma}\xi_{jk}\rangle (k = 1,2,\dots,r_j, i = 1,2,\dots,q)$. Then the system (4.1) is approximately controllable at time b if and only if $q \geq r = max\{r_j\}$ and

$$rank\, H_j = r_j \text{ for any } j = 1,2,\dots. \quad (4.9)$$

5. Examples

In this part, three different examples are treated, where the multiplicities of the eigenvalue of the operator A are one, finite and infinite, respectively.

Example 5.1: Let $\Omega = (0,1) \times (0,1)$ be an open bounded subset of $\mathbf{R}^2$ and let

$$A_1 := \triangle = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right). \quad (5.1)$$

Then the eigenvalue of A_1 are $\lambda_{n_1,n_2} = (n_1^2 + n_2^2)\pi^2$ and the corresponding eigenfunctions are Courant and Hilbert (1966) and Zerrik, Boutoulout, and El Jai (2000)

$$\xi_{n_1,n_2}(x_1,x_2) = 2\sin(n_1\pi x_1)\sin(n_2\pi x_2), \quad n_1,n_2 = 1,2,\dots. \quad (5.2)$$

In addition, we obtain that A_1 is a symmetric, uniformly elliptic operator and the multiplicity in this case is one.

In the following, we first consider the sub-diffusion equation of order $0 < \alpha < 1$ with the control inputs emerging in the differential equation as distributed inputs:

$$\begin{cases} {}_0^C D_t^\alpha z(t,x) + A_1 z(t,x) = g_1(x)u_1(t) & \text{in } [0,b] \times \Omega, \\ z(t,x) = 0 & \text{on } [0,b] \times \partial\Omega, \\ z(0,x) = 0 & \text{in } \Omega. \end{cases} \quad (5.3)$$

Since $\xi_{n_1,n_2}(x_1,x_2) = 0$ on $\partial\Omega$, it follows from Theorem 3.1 that the system (5.3) is approximately controllable if and only if

$$\int_0^1\int_0^1 g_1(x_1,x_2)\xi_{n_1,n_2}(x_1,x_2)dx_1dx_2 \neq 0, \text{ for } n_1,n_2 = 1,2,\dots. \quad (5.4)$$

More precisely, if we let $g_1(x_1,x_2) = \sum_{n_1=1}^\infty\sum_{n_2=1}^\infty \sigma_{n_1,n_2}\xi_{n_1,n_2}(x_1,x_2)$, such that $\sum_{n_1=1}^\infty\sum_{n_2=1}^\infty \sigma^2_{n_1,n_2} < \infty$. Then the system (5.3) is approximately controllable if and only if $\sigma_{n_1,n_2} \neq 0$ for all $n_1,n_2 = 1,2,\ \dots$.

Next, We examine the following system of fractional order α $(0 < \alpha < 1)$ with the control inputs appearing in the boundary conditions as boundary inputs:

$$\begin{cases} {}_0^C D_t^\alpha y(t,x) + A_1 y(t,x) = 0 & \text{in } (0,b) \times \Omega. \\ y(t,x) = h_1(x)u_1(t) & \text{on } (0,b) \times \partial\Omega. \\ y(0,x) = 0 \quad \text{in } \Omega. \end{cases} \quad (5.5)$$

Then it follows from Theorem 4.1 that the system (5.5) is approximately controllable if and only if

$$\left\langle h_1(x_1, x_2), \frac{\partial}{\partial \varsigma}\xi_{n_1,n_2}(x_1, x_2)\right\rangle \neq 0, \quad \text{for} \quad n_1, n_2 = 1, 2, \dots, \tag{5.6}$$

where $<\cdot, \cdot>$ is the inner products in space $L^2(\partial\Omega)$ and $\varsigma(x) = (\varsigma_1(x), \dots, \varsigma_n(x))$ is the outward unit normal vector to Σ at x. Moreover, let

$$h_1(x_1, 0) = h_1(x_1, 1) = h_1(0, x_2) = 0 \quad \text{and} \quad h_1(1, x_2) = \sum_{j=1}^{\infty} \sigma_j \sin(j\pi x_2) \tag{5.7}$$

with $\sum_{j=1}^{\infty} \sigma_j^2 < \infty$. Then the system (5.3) is approximately controllable if and only if $\sigma_j \neq 0$ for all $j = 1, 2, \dots$.

Example 5.2: Let $\Omega \subseteq \mathbf{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$ and consider the following problem:

$$\begin{cases} \left(-\sum_{m,n=1}^{2} \frac{\partial}{\partial x_m} a_{m,n} \frac{\partial}{\partial x_n} + V\right) u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.8}$$

where

$$a_{m,n} = a_{n,m},\ a_{m,n} \in C^1(\bar{\Omega}),\ 1 \leq i, j \leq 2 \text{ and } V \in C^1(\bar{\Omega}), V(x) \geq 0,\ x \in \bar{\Omega}$$

and there exists a constant $\mu > 0$, such that

$$\sum_{m,n=1}^{2} a_{m,n}\xi_m\xi_n \geq \mu|\xi|^2,\ x \in \bar{\Omega},\ \xi \in \mathbf{R}^2.$$

Then A_2 is a symmetric and uniformly elliptic operator. And it follows from Theorem A in Hoffmann-Ostenhof, Michor, and Nadirashvili (1999) or Theorem 1 in Hoffmann-Ostenhof, Hoffmann-Ostenhof, and Nadirashvili (1999) that

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$$

and the multiplicity of the jth eigenvalue $-\lambda_j$ for the problem (5.8) on Ω satisfies

$$r_j \leq 2j - 3 \quad \text{for} \quad j \geq 3.$$

Then, let $A_2 = \left(-\sum_{m,n=1}^{2} \frac{\partial}{\partial x_m} a_{m,n} \frac{\partial}{\partial x_n} + V\right)$. It follows that the multiplicity of eigenvalue $-\lambda_j$ is r_j, such that $r_j \leq 2j - 3$ for $j \geq 3$ and $r_j = 1$ for $j = 1, 2$ and let the corresponding eigenfunctions be $\xi_{jk}, j = 1, 2, \dots, k = 1, 2, \dots, r_j$.

We next apply Theorem 3.1 and Theorem 4.1 to this case.

From Theorem 3.1, we can conclude that the system (1.1) with A replaced by A_2 is approximately controllable if and only if $p \geq r = \max\{1, r_j\}, j \geq 3$ and

$$G_1 \neq 0, G_2 \neq 0 \text{ and } \operatorname{rank} G_j = r_j \text{ for any } j \geq 3, \tag{5.9}$$

where $r_j \leq 2j - 3, j \geq 3$ and G_j $(j = 1, 2, \dots)$ are defined in Equation (3.7).

From Theorem 4.1, we get that the system (4.1) with A replaced by A_2 is approximately controllable if and only if $q \geq r = \max\{1, r_j\}, j \geq 3$ and

$$H_1 \neq 0, H_2 \neq 0 \text{ and } \operatorname{rank} H_j = r_j \text{ for any } j \geq 3, \tag{5.10}$$

where $r_j \leq 2j - 3, j \geq 3$ and H_j $(j = 1, 2, \dots)$ are defined in Equation (4.8).

Example 5.3: For an arbitrary domain $\Omega \subseteq \mathbf{R}^n$, let $A_3 = \triangle\triangle$ with

$$\triangle = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right). \tag{5.11}$$

Then A_3 is a symmetric and uniformly elliptic operator. Consider the eigenvalue problem of $A_3 x - \lambda x = 0$ with the boundary condition

$$\triangle x = 0, \quad (\partial/\partial\varsigma)\triangle x = 0, \quad x \in \partial\Omega. \tag{5.12}$$

By Courant and Hilbert (1966), we easily get that zero occurs as an eigenvalue of infinite multiplicity. Replacing the operator A with A_3 in the systems (1.1) and (4.1), then we get that system (1.1) is not approximately controllable if $p < \infty$ and the system (4.1) is not approximately controllable if $q < \infty$, respectively.

6. Conclusion

This paper deals with the actuator characterisations to achieve approximate controllability for a class of fractional sub-diffusion equations excited by p actuators. The necessary and sufficient conditions for the controllability of two different cases are investigated, where the control inputs emerge in the differential equation as distributed inputs and as boundary inputs in the boundary conditions. The results here can be regarded as the extension of the work in Sakawa (1974) and Zerrik et al. (2000), which plays a significant role in the modelling of many field of real dynamic systems. The results could also provide some insights into the qualitative analysis of the design of fractional order controller and observer characterisations. For more information on the potential topics related to fractional order diffusion systems, we refer the readers to Ge Chen, and Kou (2015) and the references therein.

Acknowledgements

This work was supported by Chinese Universities Scientific Fund No.CUSF-DH-D-2014061 and the Natural Science Foundation of Shanghai (no. 15ZR1400800).

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by Chinese Universities Scientific Fund No.CUSF-DH-D-2014061 and the Natural Science Foundation of Shanghai [no. 15ZR1400800].

ORCID

YangQuan Chen http://orcid.org/0000-0002-7422-5988

References

Adams, E.E., & Gelhar, L.W. (1992). Field study of dispersion in a heterogeneous aquifer 2. Spatial moments analysis. *Water Resources Research, 28*, 3293–3307.

Agrawal, O.P. (2002). Solution for a fractional diffusion-wave equation defined in a bounded domain. *Nonlinear Dynamics, 29*, 145–155.

Cao, J., Chen, Y., & Li, C. (2014). Multi-UAV-based optimal crop-dusting of anomalously diffusing infestation of crops. In *Proceedings of the 2015 American Control Conference*. Chicago.

Cao, J., Li, C., & Chen, Y. (2015). Compact difference method for solving the fractional reaction-subdiffusion equation with Neumann boundary value condition. *International Journal of Computer Mathematics, 92*, 167–180.

Cao, K., Chen, Y.Q., Stuart, D., & Yue D., (2016). Cyber-physical modeling and control of crowd-pedestrian systems: A review and new framework. *IEEE/CAA Journal of Automatica Sinica, 2*(3), 334–344.

Cartea, Á., & Negrete, D. (2007). Fluid limit of the continuous-time random walk with general Lévy jump distribution functions. *Physical Review E, 76*, 041105.

Coopmans, C., Stark, B., Jensen, A., Chen, Y., & McKee, M. (2015). Cyber-physical systems enabled by small unmanned aerial vehicle. In K.P. Valavanis and G.J. Vachtsevanos (Eds.), *Handbook of Unmanned Aerial Vehicles*. Springer.

Courant, R., & Hilbert, D. (1966). *Methods of mathematical physics* (Vol. I). New York, NY: Interscience.

Dacorogna, B. (2008). *Direct methods in the calculus of variations* (2nd ed.). New York, NY: Springer.

El Jai, A., & Pritchard, A.J. (1988). *Sensors and controls in the analysis of distributed systems*. New York, NY: Halsted Press.

Fujishiro, K. (2015). Approximate controllability for fractional diffusion equations by Dirichlet boundary control. Retrieved from http://arxiv.org/abs/1404.0207.

Fujishiro, K., & Yamamoto, M. (2014). Approximate controllability for fractional diffusion equations by interior control. *Applied Analysis, 93*, 1793–1810.

Ge, F., Chen, Y.Q., & Kou, C. (2015). Cyber-physical systems as general distributed parameter systems: Three types of fractional order models and emerging research opportunities [J]. *IEEE/CAA Journal of Automatica Sinica, 2*(4), 353–357.

Hoffmann-Ostenhof, M., Hoffmann-Ostenhof, T., & Nadirashvili, N. (1999). On the multiplicity of eigenvalues of the Laplacian on surfaces. *Annals of Global Analysis and Geometry, 17*, 43–48.

Hoffmann-Ostenhof, T., Michor, P.W., & Nadirashvili, N. (1999). Bounds on the multiplicity of eigenvalues for fixed membranes. *Geometric and Functional Analysis, 9*, 1169–1188.

Kilbas, A.A., Srivastava, H.M., & Trujillo, J.J. (2006). *Theory and applications of fractional differential equations*. Elsevier.

Kilbas, A.A., & Trujillo, J.J. (2002). Differential equations of fractional order: Methods, results and problems, II. *Applied Analysis, 81*, 435–493.

Mandelbrot, B.B. (1983). *The fractal geometry of nature* (Vol. 173). Macmillan.

Meerschaert, M.M., & Scalas, E. (2006). Coupled continuous time random walks in finance. *Physics A, 370*, 114–118.

Metzler, R., & Klafter, J. (2000). The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Physics Reports, 339*, 1–77.

Metzler, R., & Klafter, J. (2004). The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *Journal of Physics A, 37*, 161–208.

Monje, C.A., Chen, Y.Q., Vinagre, B., Xue, D., & Liu, V. (2010). *Fractional order systems and controls-fundamentals and applications*. Springer-Verlag.

Montroll, E.W., & Weiss, G.H. (1965). Random walks on lattices. II. *Journal of Mathematical Physics, 6*, 167–181.

Podlubny, I. (1999). *Fractional differential equations*. San Diego, CA: Academic Press.

Renardy, M., & Rogers, R.C. (2006). *An introduction to partial differential equations* (Vol. 13). Springer Science & Business Media.

Sakamoto, K., & Yamamoto, M. (2011). Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *Journal of Mathematical Analysis and Applications, 382*, 426–447.

Sakawa, Y. (1974). Controllability for partial differential equations of parabolic type. *SIAM Journal on Control, 12*, 389–400.

Sario, L., & Weill, G. (1965). Normal operators and uniformly elliptic self-adjoint partial differential equations. *Transactions of the American Mathematical Society*, 225–235.

Spears, W.M., & Spears, D.F. (2012). *Physicomimetics: Physics-based swarm intelligence*. Springer Science & Business Media.

Torvik, P.J., & Bagley, R.L. (1984). On the appearance of the fractional derivative in the behavior of real materials. *Journal of Applied Mechanics, 51*(2), 294–298.

Uchaikin, V., & Sibatov, R. (2013). *Fractional kinetics in solids: Anomalous charge transport in semiconductors*. World Scientific Publishing.

Zerrik, E., Boutoulout, A., & El Jai, A. (2000). Acutators and regional boundary controllability for parabolic systems. *International Journal of Systems Science, 31*, 73–82.

Appendix The proof of Lemma 4.2

Proof: Decompose z into

$$y(t,x)=\omega(t,x)+\sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{q}\left(u_{h_i},\xi_{jk}\right)\xi_{jk}(x)u_i(t),$$

such that $u_{h_i}(x)u_i(t)=0$ on $(0,b)\times\Sigma$ for any $i=1,2,\dots,q$ and ω is the solution of the following problem:

$$\begin{cases} {}_0^CD_t^{\alpha}\omega(t,x)+A\omega(t,x) & \text{in } (0,b)\times\Omega,\\ \quad =-\sum\limits_{i=1}^{q}u_{h_i}(x){}_0^CD_t^{\alpha}u_i(t) & \\ \omega(t,x)=0 & \text{on } (0,b)\times\Sigma,\\ \omega(0,x)=0 & \text{in } \Omega. \end{cases} \tag{A.1}$$

We see that $y(t,x)=\omega(t,x)+\sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{q}\left(u_{h_i},\xi_{jk}\right)\xi_{jk}(x)u_i(t)$ satisfies (4.1). In addition, it follows from Theorem 2.2 in Sakamoto and Yamamoto (2011) that ω can be given by

$$\omega(t,x)=-\sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\left(\int_0^t\left(\sum_{i=1}^{q}u_{h_i}(x){}_0^CD_t^{\alpha}u_i(t),\xi_{jk}\right)\right.$$
$$\left.\times(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-\tau)^{\alpha})\,d\tau\right)\xi_{jk}(x).$$

Then by Lemma 2.1 and Lemma 2.2, one has

$$\omega(t,x)$$
$$=-\sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{q}\left(\int_0^t{}_0I_t^{1-\alpha}\left(\frac{\partial}{\partial t}u_i(t)\right)(t-\tau)^{\alpha-1}\right.$$
$$\left.\times E_{\alpha,\alpha}(-\lambda_j(t-\tau)^{\alpha})d\tau\right)(u_{h_i},\xi_{jk})\xi_{jk}(x)$$
$$=-\sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{q}\left({}_0I_t^{1-\alpha}\left(\frac{\partial}{\partial t}u_i(t)\right)*\right.$$
$$\left.\times\left(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_jt^{\alpha})\right)\right)(u_{h_i},\xi_{jk})\xi_{jk}(x)$$
$$=-\sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{q}\left(\frac{\partial}{\partial t}u_i(t)*{}_0I_t^{1-\alpha}\left(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_jt^{\alpha})\right)\right)$$
$$\times(u_{h_i},\xi_{jk})\xi_{jk}(x)$$
$$=-\sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{q}\left(\int_0^t\frac{\partial}{\partial t}u_i(t-\tau){}_0I_t^{1-\alpha}\right.$$
$$\left.\times\left(\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j\tau^{\alpha})\right)d\tau\right)(u_{h_i},\xi_{jk})\xi_{jk}(x)$$
$$=\sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{q}\left(\int_0^t\frac{\partial}{\partial\tau}u_i(t-\tau)E_{\alpha}(-\lambda_j\tau^{\alpha})d\tau\right)$$
$$\times(u_{h_i},\xi_{jk})\xi_{jk}(x).$$

Since $u_{h_i}(x)u_i(t)=0$ on $(0,b)\times\Sigma$, the integration by parts gives

$$\omega(t,x)=\sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{q}\left(-u_i(t)-\int_0^t u_i(t-\tau)\frac{\partial}{\partial\tau}E_{\alpha}(-\lambda_j\tau^{\alpha})d\tau\right)$$
$$\times(u_{h_i},\xi_{jk})\xi_{jk}(x). \tag{A.2}$$

By Lemma 4.1, Equations (2.2) and (A.2), we have

$$y(t,x)=\omega(t,x)+\sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{q}(u_{h_i},\xi_{jk})\xi_{jk}(x)u_i(t)$$
$$=\sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{q}\int_0^t\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j\tau^{\alpha})$$
$$\times u_i(t-\tau)d\tau\lambda_j(u_{h_i},\xi_{jk})\xi_{jk}(x)$$
$$=\sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{q}\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-\tau)^{\alpha})$$
$$\times u_i(\tau)d\tau\left\langle h_i,\frac{\partial}{\partial\varsigma}\xi_{jk}\right\rangle\xi_{jk}(x).$$

In addition, similar to the proof of Theorem 2.1 in Sakamoto and Yamamoto (2011), we obtain the uniqueness of the mild solution to the problem (4.1). This completes the proof. □