J. Math. Anal. Appl. 440 (2016) 865–884

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Regional gradient controllability of sub-diffusion processes

Fudong Ge [a], YangQuan Chen [b,*], Chunhai Kou [c]

[a] College of Information Science and Technology, Donghua University, Shanghai 201620, PR China
[b] Mechatronics, Embedded Systems and Automation Lab, University of California, Merced, CA 95343, USA
[c] Department of Applied Mathematics, Donghua University, Shanghai 201620, PR China

ARTICLE INFO

Article history:
Received 15 October 2015
Available online 31 March 2016
Submitted by P. Yao

Keywords:
Regional gradient controllability
Minimum energy
Fractional order
Sub-diffusion process
Strategic actuators

ABSTRACT

By sub-diffusion systems we mean to replace the first order time derivative of normal diffusion system with a Riemann–Liouville time fractional derivative of order $\alpha \in (0, 1)$. In this paper, we first introduce a new definition of regional gradient controllability for the sub-diffusion system with the control inputs appearing in the differential equations as distributed inputs, which recovers the usual definition of regional gradient controllability as $\alpha \to 1$. The sufficient and necessary conditions on actuators to achieve the gradient controllability in a given subregion of the whole domain are presented. An approach to guarantee the regional gradient controllability of the problems studied within the considered subregion using minimum energy control effort is presented. Several examples are presented in the end to illustrate the effectiveness of our results, where zone actuators, pointwise actuators or filament actuators are respectively considered.

1. Introduction

It is well known that the diffusive transport is one of the most important transport mechanisms found in our real-life. Recently, after the first introduction of the continuous time random walk (CTRW) by Montroll and Weiss in [21], a growing number of contributions have shown the prevalence of diffusion in which the mean squared displacement is smaller (in the case of sub-diffusion) or bigger (in the case of super-diffusion) than that in a Gaussian diffusion process. More precisely, when the particles are assumed to jump at fixed time intervals with an incorporating waiting times, the particles will experience the sub-diffusion process and when the particles are supposed to jump following from a general, non-Gaussian jump distribution function, the particles will undergo the super-diffusion process [2,11,29]. Unlike the typical diffusion, the mean squared displacement (MSD) of anomalous diffusion process (sub-diffusion process or super-diffusion

* Corresponding author.
E-mail address: ychen53@ucmerced.edu (Y. Chen).

http://dx.doi.org/10.1016/j.jmaa.2016.03.051

process) is described by a power law of fractional exponent [30,18]. The fractional diffusion systems provide a natural description of non-local transport, well characterizing those anomalous processes [20,19].

It is worth noting that in the case of diffusion systems, in general, not all the states can be reached in the whole domain of interest [6,1]. Let Ω be a connected, open bounded subset of $\mathbf{R}^n$ with Lipschitz continuous boundary $\partial\Omega$ and denote $Q = \Omega \times [0, b]$, $\Sigma = \partial\Omega \times [0, b]$, here we investigate the following time fractional order diffusion system:

$$\begin{cases} {}_0D_t^\alpha y(x,t) + Ay(x,t) = Bu(t) \text{ in } Q, \\ y(\eta,t) = 0 \text{ on } \Sigma, \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} y(x,t) = y_0(x) \text{ in } \Omega, \end{cases} \tag{1.1}$$

where ${}_0D_t^\alpha$ and ${}_0I_t^\alpha$ denote the Riemann–Liouville fractional order derivative and integral with respect to time t, respectively, given by [23,13]

$${}_0D_t^\alpha y(x,t) = \frac{\partial}{\partial t} {}_0I_t^{1-\alpha} y(x,t), \ 0 < \alpha < 1 \tag{1.2}$$

and

$${}_0I_t^\alpha y(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(x,s)ds, \ \alpha > 0, \tag{1.3}$$

A is a uniformly elliptic operator and for more properties on A, we refer the readers to [26,28,31], and $y_0 \in Y := H_0^1(\Omega)$. Moreover, since $\Omega \subseteq \mathbf{R}^n$ is bounded, the Poincaré inequality [12]

$$\int_\Omega |f|^2 \le C^2 \int_\Omega |\nabla f|^2, \quad f \in H_0^1(\Omega), \ C = C(\Omega) \text{ is a constant} \tag{1.4}$$

states that the space $H_0^1(\Omega)$ with the inner product

$$(f_1, f_2) = (\nabla f_1, \nabla f_2)_{L^2(\Omega)}, \quad f_i \in H_0^1(\Omega), \ i = 1, 2 \tag{1.5}$$

is a Hilbert space. In addition, $B : \mathbf{R}^p \to Y$ is the control operator depending on the number and the structure of actuators, the control $u \in L^2(0, b; \mathbf{R}^p)$ and p is the number of the actuators. See [6,5] for more information on how to use the sensors and actuators to investigate the structures and properties of distributed parameter systems of integer order.

When $\alpha = 1$, the problem (1.1) reduces to the classical integer order diffusion system and the regional gradient controllability of which, regarded as a particular case here, was studied in [32,33]. Moreover, as we all know, in many real-world dynamic systems, this regional idea occurs naturally when one is interested in steering the system to the states in a subregion of the spatial domain. For more information on the regional controllability idea, we refer the reader to [6,1] and the references cited therein. In particular, we point out that the concept of regional gradient controllability is both practical and beneficial to study those non-gradient-controllable systems in the whole domain. This is due to the fact that we may be only interested in the knowledge of the states in a critical subregion of the whole domain.

Motivated by the above discussions, in this paper, our goal is to study the regional gradient controllability of the system (1.1) with minimum energy control inputs and explore the characteristics of the strategic actuators. To the best of our knowledge, no result is available on this topic. We hope that the results here could provide some insights into the control theory of this field and be used in real-life application. The remainder contents of this paper are structured as follows. Some preliminary results are introduced in the next section and in Section 3, the characteristics of the gradient strategic actuators are investigated. An approach steering the system (1.1) from the initial gradient vector to a target gradient function in the given subregion with minimum energy control is presented in Section 4. Several illustrative application examples are given in the end of this paper followed by conclusions.

2. Preliminary results

In this section, we start by introducing some preliminary results to be used thereafter.

Based on the Laplace transform of the Riemann–Liouville time fractional derivative of order $\alpha \in (0,1)$:

$$\mathcal{L}[{}_0D_t^\alpha y(x,t)] = s^\alpha \mathcal{L}y(x,s) - y_0(x), \tag{2.1}$$

similar to the argument in [27,8], then the system (1.1) admits a unique mild solution given by

$$y(x,t) = \sum_{j=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^\alpha)\,(\xi_j, y_0)\,\xi_j(x) + \sum_{j=1}^{\infty} \int_0^t \frac{E_{\alpha,\alpha}(-\lambda_j (t-\tau)^\alpha)}{(t-\tau)^{1-\alpha}}\,(\xi_j, Bu(\tau)) d\tau \xi_j(x), \tag{2.2}$$

where $\{\lambda_j\}_{j=1,2,\cdots}$ with

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots, \quad \lim_{j\to\infty} \lambda_j = \infty$$

are the eigenvalues of A, ξ_j is the corresponding orthonormal eigenfunction of λ_j in $H_0^1(\Omega)$ [3] and

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \ \ \mathbf{Re}\,\alpha > 0,\ \beta, z \in \mathbf{C}$$

is known as the generalized Mittag-Leffler function in two parameters. Moreover, since [10]

$$|E_{\alpha,\alpha}(-\lambda_j t^\alpha)| \le 1, \quad \alpha \in (0,1],\ t \in [0,b],\ \lambda_j > 0,\ j = 1,2,\cdots,$$

it then follows from $y_0 \in H_0^1(\Omega)$ and $B : \mathbf{R}^p \to H_0^1(\Omega)$ that $y(x,\cdot) \in H_0^1(\Omega)$ for any fixed $t \in [0,b]$.

Let $\omega \subseteq \Omega$ be a given region of positive Lebesgue measure and suppose that $y_b = y(\cdot,b) \in H_0^1(\Omega)$ with $y(x,t)$ satisfying (2.2), then the regional gradient controllability problem is concerned with whether there exists a control u to steer the system (1.1) from the initial gradient vector ∇y_0 to any gradient vector in $\left(L^2(\omega)\right)^n$.

Let $\nabla : H_0^1(\Omega) \to \left(L^2(\Omega)\right)^n$ be the operator defined by

$$y \to \nabla y := \left(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \cdots, \frac{\partial y}{\partial x_n}\right). \tag{2.3}$$

By [15], the adjoint of the gradient operating on a connected, open bounded subset Ω with a Lipschitz continuous boundary $\partial\Omega$ is the minus of the divergence operator. Then $\nabla^* : \left(L^2(\Omega)\right)^n \to H^{-1}(\Omega)$ is given by

$$\xi \to \nabla^* \xi := v \tag{2.4}$$

and v solves the following Dirichlet problem

$$\begin{cases} v = -\operatorname{div}(\xi) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Consider the following two restriction mappings

$$p_\omega : \begin{cases} \left(L^2(\Omega)\right)^n \to \left(L^2(\omega)\right)^n, \\ \xi \to \xi|_\omega \end{cases} \quad \text{and} \quad p_{1\omega} : \begin{cases} L^2(\Omega) \to L^2(\omega), \\ y \to y|_\omega. \end{cases} \tag{2.6}$$

Their adjoint operators are, respectively, denoted by

$$p_\omega^* : \begin{cases} \left(L^2(\omega)\right)^n \to \left(L^2(\Omega)\right)^n, \\ \xi \to p_\omega^* \xi = \begin{cases} \xi, & x \in \omega, \\ 0, & x \in \Omega \backslash \omega \end{cases} \end{cases} \tag{2.7}$$

and

$$p_{1\omega}^* : \begin{cases} L^2(\omega) \to L^2(\Omega), \\ y \to p_{1\omega}^* y = \begin{cases} y, & x \in \omega, \\ 0, & x \in \Omega \backslash \omega. \end{cases} \end{cases} \tag{2.8}$$

In order to state the main results, the following two assumptions are supposed to hold all over the article:

(A_1) B is a densely defined operator and B^* exists.
(A_2) $(BK_\alpha(t))^*$ exists and $(BK_\alpha(t))^* = K_\alpha^*(t)B^*$.

In particular, when B is bounded, it is easy to see that (A_1) and (A_2) hold.

Moreover, for the integer order derivative, as we all know, the following result on integration by part holds:

$$\textstyle\int_a^b f(t)g'(t)dt = [f(t)g(t)]_{t=a}^{t=b} - \int_a^b g(t)f'(t)dt. \tag{2.9}$$

For the fractional order cases, however, the Eq. (2.9), even the first order derivative being replaced by a Riemann–Liouville fractional order derivative or Caputo fractional order derivative of order $\alpha \in (0,1)$ will never hold. Fortunately, by [24], we have the following lemma.

Lemma 2.1. *(See [24].) For any $t \in [a,b]$ and α $(0 < \alpha < 1)$, the following formula holds*

$$\textstyle\int_a^b f(t)\,{}_aD_t^\alpha g(t)dt = \left[f(t)\,{}_aI_t^{1-\alpha} g(t)\right]_{t=a}^{t=b} - \int_a^b g(t)\,{}_t^C D_b^\alpha f(t)dt, \tag{2.10}$$

where ${}_t^C D_b^\alpha$ denotes the right-sided Caputo fractional order derivative with respect to time t of order $\alpha \in (0,1]$ given by [23,13,24]

$${}_t^C D_b^\alpha \rho(x,t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^b (\tau - t)^{-\alpha} \frac{\partial}{\partial \tau} \rho(x,\tau) d\tau. \tag{2.11}$$

Then the duality relationship between gradient controllability and gradient observability of the linear fractional order system may be lost. Next, we shall introduce a new definition of the regional gradient controllability for the system (1.1), which recovers the usual definition of regional gradient controllability as $\alpha \to 1$. So it can be regarded as an extension of the existence contributions.

Definition 2.1. The system (1.1) is called to be regionally exactly (respectively, approximately) gradient controllable in ω at time b if for any gradient vector $d(x) \in \left(L^2(\omega)\right)^n$, given $\varepsilon > 0$, there exists a control $u \in L^2(0, b; \mathbf{R}^p)$ such that $p_\omega \nabla \left[{}_0I_b^{1-\alpha} y(x,b)\right] = d(x)$ (respectively, $\|p_\omega \nabla \left[{}_0I_b^{1-\alpha} y(x,b)\right] - d(x)\|_{(L^2(\omega))^n} \le \varepsilon$).

Further, taking into account that (1.1) is a line system, by the Proposition 3.1 in [9], it suffices to suppose that $y_0 = 0$ in the following discussion.

Since $y_b = y(\cdot, b) \in Y$, let $H : L^2(0, b; \mathbf{R}^p) \to Y$ be

$$Hu = {}_0I_b^{1-\alpha} y(x,b), \ \forall u \in L^2(0, b; \mathbf{R}^p). \tag{2.12}$$

Then we see

$$\begin{aligned} Hu &= \int_0^b \sum_{j=1}^\infty \int_0^s \tfrac{E_{\alpha,\alpha}(-\lambda_j(s-\tau)^\alpha)(\xi_j, Bu(\tau))}{\Gamma(1-\alpha)(b-s)^\alpha(s-\tau)^{1-\alpha}} d\tau ds \xi_j(x) \\ &= \sum_{j=1}^\infty \int_0^b \int_\tau^b \sum_{n=0}^\infty \tfrac{(-\lambda_j)^n (s-\tau)^{\alpha n+\alpha-1}(\xi_j, Bu(\tau))}{\Gamma(\alpha n+\alpha)\Gamma(1-\alpha)(b-s)^\alpha} ds d\tau \xi_j(x) \\ &= \sum_{j=1}^\infty \int_0^b E_\alpha(-\lambda_j(b-\tau)^\alpha)\,(\xi_j, Bu(\tau)) d\tau \xi_j(x), \end{aligned}$$

where

$$E_\alpha(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n+1)}, \ \mathbf{Re}\,\alpha > 0, \ z \in \mathbf{C}$$

is known as the Mittag-Leffler function in one parameter. For any $v \in Y^*$, it follows from the property of duality relationship $\langle Hu, v\rangle_{Y\times Y^*} = \langle u, H^*v\rangle_{L^2(0,b;\mathbf{R}^p)\times L^2(0,b;\mathbf{R}^p)}$ that

$$(H^*v)(t) = B^* \sum_{j=1}^\infty E_\alpha(-\lambda_j(b-t)^\alpha)\,(\xi_j, v)\,\xi_j(x), \tag{2.13}$$

where B^* is the adjoint operator of B.

We now are ready to state the following two propositions.

Proposition 2.1. *There is an equivalence among the following properties:*

(1) *The system (1.1) is regionally exactly gradient controllable on ω at time b;*
(2) *$Im(p_\omega \nabla H) = \left(L^2(\omega)\right)^n$;*
(3) *$Ker(p_\omega) + Im(\nabla H) = \left(L^2(\Omega)\right)^n$;*
(4) *There exists a constant $c > 0$ such that for all $y \in \left(L^2(\omega)\right)^n$,*

$$\|y\|_{(L^2(\omega))^n} \le c\|H^*\nabla^* p_\omega^* y\|_{L^2(0,b;\mathbf{R}^p)}. \tag{2.14}$$

Proof. By Definition 2.1, (1) $\Leftrightarrow$ (2). Next, we show (2) $\Leftrightarrow$ (3).

(2) $\Rightarrow$ (3): For any $y \in \left(L^2(\omega)\right)^n$, assume that $\hat{y}$ is the extension of y in $\left(L^2(\Omega)\right)^n$. By (2) and the definition of mapping p_ω, there exists a $u \in L^2(0, b; \mathbf{R}^p)$ such that $y_1 \in Ker(p_\omega)$ and $\hat{y} = y_1 + Hu$.

(3) $\Rightarrow$ (2): For any $\tilde{y} \in \left(L^2(\Omega)\right)^n$, from (3), $\tilde{y} = y_1 + y_2$, where $y_1 \in Ker(p_\omega)$ and $y_2 \in Im(\nabla H)$. Then there exists a $u \in L^2(0, b; \mathbf{R}^p)$ such that $\nabla Hu = y_2$. Hence, it follows from the definition of p_ω that (2) holds.

(1) $\Leftrightarrow$ (4): Consider the following results in [25]:

Let X_1, X_2, X_3 be reflexive Hilbert spaces, $f_1 \in \mathcal{L}(X_1, X_3)$ be a continuous bounded operator from X_1 to X_3 and $f_2 \in \mathcal{L}(X_2, X_3)$ be a continuous bounded operator from X_2 to X_3. Then the following two properties are equivalent

a) $Im(f_1) \subseteq Im(f_2)$
b) $\exists\, c > 0$ such that $\|f_1^* x^*\|_{X_1^*} \leq c\|f_2^* x^*\|_{X_2^*}$, $\forall x^* \in X_3^*$,

where f_1^* and f_2^* are, respectively, the adjoint operators of f_1 and f_2, X_1^*, X_2^* and X_3^* are, respectively, the duality spaces of X_1, X_2 and X_3.

By choosing $X_1 = X_3 = \big(L^2(\omega)\big)^n$, $X_2 = L^2(0, b; \mathbf{R}^p)$, $f_1 = Id_{(L^2(\omega))^n}$ and $f_2 = p_\omega \nabla H$, we then complete the proof. □

Proposition 2.2. *The following properties are equivalent:*

$\langle 1\rangle$ *The system (1.1) is regionally approximately gradient controllable on ω at time b;*
$\langle 2\rangle$ $\overline{Im(p_\omega \nabla H)} = \big(L^2(\omega)\big)^n$;
$\langle 3\rangle$ $Ker(p_\omega) + \overline{Im(\nabla H)} = \big(L^2(\Omega)\big)^n$;
$\langle 4\rangle$ *The operator $p_\omega \nabla H H^* \nabla^* p_\omega^*$ is positive definite.*

Proof. From Proposition 2.1, $\langle 1\rangle \Leftrightarrow \langle 2\rangle \Leftrightarrow \langle 3\rangle$ and we only need to show that $\langle 2\rangle \Leftrightarrow \langle 4\rangle$. Indeed, since

$$\overline{Im(p_\omega \nabla H)} = \big(L^2(\omega)\big)^n \Leftrightarrow (p_\omega \nabla H u, y)_{(L^2(\omega))^n} = 0, \forall u \in L^2(0, b; \mathbf{R}^p) \text{ implies } y = 0.$$

Without loss of generality, let $u = H^* \nabla^* p_\omega^* y$, $y \in \big(L^2(\omega)\big)^n$. This then allows us to complete the proof. □

Remark 2.1. 1) In [33], the particular cases of the system (1.1) with $\alpha = 1$ are considered.

2) Consider the set

$$W_\omega = \big\{u \in L^2(0, b; \mathbf{R}^p) : p_{1\omega} H u \in L^2(\omega),\ \omega \subseteq \Omega\big\} \tag{2.15}$$

and the cost functional

$$J(u) = \int_0^b \|u(t)\| dt, \tag{2.16}$$

we then have $W_{\omega_2} \subseteq W_{\omega_1}$ for any $\omega_1 \subseteq \omega_2 \subseteq \Omega$ and consequently

$$\min_{u \in W_{\omega_1}} J(u) \leq \min_{u \in W_{\omega_2}} J(u). \tag{2.17}$$

Next, we will present a example, which is regionally gradient controllable on a subregion $\omega \subseteq \Omega$ but not gradient controllable on the whole domain Ω.

Example 2.1. Let $\Omega = [0, 1] \times [0, 1] \subseteq \mathbf{R}^2$ and consider the following sub-diffusion system excited by a filament actuator:

$$\begin{cases} {}_0D_t^\alpha y(x_1, x_2, t) - \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) y(x_1, x_2, t) = f(x)u(t) \ \text{ in } Q, \\ y(\eta_1, \eta_2, t) = 0 \ \text{ on } \Sigma, \\ \lim\limits_{t \to 0^+} {}_0I_t^{1-\alpha} y(x_1, x_2, t) = 0 \ \text{ in } \Omega, \end{cases} \tag{2.18}$$

where $Bu(t) = f(x)u(t) = \delta(x_1 - \frac{1}{2})\sin(\pi x_2)u(t)$ and $\delta(x)$ is a Dirac delta function on the real number line that is zero everywhere except at zero.

According to (1.1), we get that $A = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$, with the eigenvalues being $\lambda_{ij} = (i^2 + j^2)\pi^2$ and the corresponding eigenfunctions being $\xi_{ij}(x_1, x_2) = 2\sin(i\pi x_1)\sin(j\pi x_2)$ [22]. It then follows from Eq. (2.2), Eq. (2.12) and Eq. (2.13) that

$$Hu = \sum_{i,j=1}^{\infty} \int_0^b E_\alpha(-\lambda_{ij}(b-\tau)^\alpha)\,(\xi_{ij}, Bu(\tau))d\tau\xi_{ij}(x) \tag{2.19}$$

and

$$(H^*v)(t) = B^* \sum_{i,j=1}^{\infty} E_\alpha(-\lambda_{ij}(b-t)^\alpha)\,(\xi_{ij}, v)\,\xi_{ij}(x). \tag{2.20}$$

Let

$$d = (\cos(\pi x_1)\sin(3\pi x_2), 3\sin(\pi x_1)\cos(3\pi x_2)) \in \left(L^2(\Omega)\right)^2.$$

By the definition of ∇^* in (2.4), one has

$$\nabla^* d = 10\pi\sin(\pi x_1)\sin(3\pi x_2).$$

Moreover, since $B : \mathbf{R} \to Y$, for any t fixed and any $v_1(t) \in Y^*$, it follows from the duality relationship $\langle Bu(t), v_1(t)\rangle_{Y\times Y^*} = \langle u(t), B^*v_1(t)\rangle_{\mathbf{R}\times\mathbf{R}}$ that

$$(B^*v_1)(t) = -\int_0^1\int_0^1 \delta(x_1 - \frac{1}{2})\sin(\pi x_2)\triangle v_1(t)dx_1dx_2. \tag{2.21}$$

Then

$$\begin{aligned}(H^*\nabla^*d)(t) &= -c\sum_{i,j=1}^{\infty} E_\alpha(-\lambda_{ij}(b-t)^\alpha)\,(\nabla^*d, \xi_{ij})\sin\left(\tfrac{i\pi}{2}\right)\int_0^1 \sin(\pi x_2)\sin(j\pi x_2)dx_2 \\ &= -10\pi c\sum_{i,j=1}^{\infty} E_\alpha(-\lambda_{ij}(b-t)^\alpha)\int_0^1 \sin(\pi x_1)\sin(i\pi x_1)dx_1 \\ &\quad\times \int_0^1 \sin(3\pi x_2)\sin(j\pi x_2)dx_2\sin\left(\tfrac{i\pi}{2}\right)\int_0^1 \sin(\pi x_2)\sin(j\pi x_2)dx_2 \\ &= 0\end{aligned}$$

and $c = 2(i^2\pi^2 + j^2\pi^2)$. However, let $\omega = [0,1] \times [0, 1/6]$, we have

$$\begin{aligned}(H^*\nabla^*p_\omega^*p_\omega d)(t) &= -10\pi c\sum_{i,j=1}^{\infty} E_\alpha(-\lambda_{ij}(b-t)^\alpha)\int_0^1 \sin(\pi x_1)\sin(i\pi x_1)dx_1 \\ &\quad\times \int_0^{1/6} \sin(3\pi x_2)\sin(j\pi x_2)dx_2\sin\left(\tfrac{i\pi}{2}\right)\int_0^1 \sin(\pi x_2)\sin(j\pi x_2)dx_2 \\ &= -\tfrac{5\sqrt{3}c}{32}E_\alpha(-2\pi^2(b-t)^\alpha) \neq 0,\end{aligned}$$

which means that d is exactly gradient controllable in ω but which is not exactly gradient controllable on the whole domain Ω.

The following lemma plays a significant role to obtain our results.

Lemma 2.2. *(See [4].) Let $\Omega \subseteq \mathbf{R}^n$ be an open set and $C_0^\infty(\Omega)$ be the class of infinitely differentiable functions in ω at time b with compact support in Ω and $u \in L^1_{loc}(\Omega)$ be such that*

$$\int_\Omega u(x)\psi(x)dx = 0, \quad \forall \psi \in C_0^\infty(\Omega). \tag{2.22}$$

Then $u = 0$ almost everywhere in Ω.

3. Regional strategic actuators

In this section, we focus on addressing the link between regionally approximately gradient controllable and actuators structure.

Consider the system (1.1) and suppose that it is excited by p actuators $(D_i, f_i)_{1\le i\le p}$, where $D_i \subseteq \Omega$ is the support of the actuator and $f_i \in Y$ is its spatial distribution, $i = 1, 2, \cdots, p$:

$$\begin{cases} {}_0D_t^\alpha y(x,t) + Ay(x,t) = \sum\limits_{i=1}^{p} p_{D_i} f_i(x) u_i(t) \ \text{ in } \ Q, \\ y(\eta, t) = 0 \ \text{ on } \ \Sigma, \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} y(x,t) = 0 \ \text{ in } \ \Omega. \end{cases} \tag{3.1}$$

As for actuator (D, f), for example, if $D = \{\sigma\}$ with $\sigma \in \bar{\Omega}$ and $f = \delta_\sigma$, where $\delta_\sigma = \delta(\cdot - \sigma)$ is a generalized function in ω at time b that is zero everywhere except at σ, the actuator is called pointwise actuator and in this case, $B : \mathbf{R}^p \to L^2(\Omega)$ may be unbounded. It is called zone actuator when $D \subseteq \bar{\Omega}$ and $f \in Y$ with $B \in \mathcal{L}\left(\mathbf{R}^p, L^2(\Omega)\right)$ is bounded. For more knowledge on the properties of actuators and sensors, we refer the readers to [6,5].

Definition 3.1. A actuator (or a suite of actuators) is said to be gradient ω-strategic at time b if the system studied is regionally approximately gradient controllable in ω at time b.

Since A is a uniformly elliptic operator, for any $y_i \in L^2(0, b; L^2(\Omega))$, $i = 1, 2$, we have

$$\begin{aligned} &\int_Q y_1(x,t)Ay_2(x,t)dtdx - \int_Q y_2(x,t)A^* y_1(x,t)dtdx \\ = &\int_{\partial\Omega\times[0,b]} \left[y_1(\eta,t)\frac{\partial y_2(\eta,t)}{\partial v_A} - y_2(\eta,t)\frac{\partial y_2(\eta,t)}{\partial v_{A^*}} \right] dtd\eta. \end{aligned}$$

Moreover, by [3], each eigenvalue λ_j $(j = 1, 2, \cdots)$ has a multiplicity r_j such that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots, \quad \lim_{j\to\infty} \lambda_j = \infty.$$

Moreover, for each $j = 1, 2, \cdots$, ξ_{jk} $(k = 1, 2, \cdots, r_j)$ is supposed to be the orthonormal eigenfunction corresponding to λ_j such that

$$(\xi_{jk_m}, \xi_{jk_n}) = \begin{cases} 1, & k_m = k_n, \\ 0, & k_m \ne k_n, \end{cases}$$

where $1 \le k_m, k_n \le r_j$, $k_m, k_n \in \mathbf{N}$. Then the sequence $\{\xi_{jk}, k = 1, 2, \cdots, r_j, j = 1, 2, \cdots\}$ is an orthonormal basis in Y and for any $y(x) \in Y$, one has

$$y(x) = \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} (y, \xi_{jk}) \xi_{jk}(x).$$

Theorem 3.1. *Given $b > 0$, for any $j = 1, 2, \cdots,\ s = 1, 2, \cdots, n$, define the following $p \times r_j$ matrices G_j^s*

$$G_j^s = \begin{bmatrix} \xi_{j1}^{1s} & \xi_{j2}^{1s} & \cdots & \xi_{jr_j}^{1s} \\ \xi_{j1}^{2s} & \xi_{j2}^{2s} & \cdots & \xi_{jr_j}^{2s} \\ \vdots & \vdots & \vdots & \vdots \\ \xi_{j1}^{ps} & \xi_{j2}^{ps} & \cdots & \xi_{jr_j}^{ps} \end{bmatrix}_{p \times r_j}, \tag{3.2}$$

where $\xi_{jk}^{is} = \left(\frac{\partial \xi_{jk}}{\partial x_s}, f_i \right)_{L^2(D_i)}$, $i = 1, 2, \ldots, p$ and $k = 1, 2, \cdots, r_j$. Let $z_{jks} = (p_{1\omega}^* z_s, \xi_{jk})$, $z_{js} = (z_{j1s}, z_{j2s}, \cdots, z_{jr_js})^T \in \mathbf{R}^{r_j}, \mathbf{0}_p = (0, 0, \cdots, 0) \in \mathbf{R}^p$ and $\mathbf{0}_n = (0, 0, \cdots, 0) \in \left(L^2(\omega)\right)^n$. Then the actuators $(D_i, f_i)_{1 \le i \le p}$ are gradient ω-strategic at time b if and only if for all $j = 1, 2, \cdots$,

$$\sum_{s=1}^{n} G_j^s z_{js} = \mathbf{0}_p \ \Rightarrow z = \mathbf{0}_n \ \text{in} \ \left(L^2(\omega)\right)^n. \tag{3.3}$$

In particular, when $n = 1$, (3.3) is equivalent to

$$(1)\ p \ge r = \max\{r_j\}; \qquad (2)\ \mathit{rank}\, G_j^1 = r_j \ \mathit{for\ all}\ j = 1, 2, \cdots.$$

Proof. By Definition 2.1, the actuators $(D_i, f_i)_{1 \le i \le p}$ are gradient ω-strategic at time b if and only if

$$\overline{Im(p_\omega \nabla H)} = \left(L^2(\omega)\right)^n \ (\text{i.e.,}\ \ Ker(H^* \nabla^* p_\omega^*) = \{\mathbf{0}_n\}), \tag{3.4}$$

where H is defined by (2.12) with

$$Bu = \sum_{i=1}^{p} p_{D_i} f_i(x) u_i(t).$$

This allows us to say that the necessary and sufficient condition for the gradient ω-strategic of the actuators $(D_i, f_i)_{1 \le i \le p}$ at time b is that

$$\left\{ \begin{array}{l} z \in \left(L^2(\omega)\right)^n : (p_\omega \nabla H u, z) = 0, \\ \quad \forall u \in L^2(0, b; \mathbf{R}^p) \end{array} \right\} \Rightarrow z = \mathbf{0}_n, \tag{3.5}$$

where

$$z = (z_1, z_2, \cdots, z_n) \in \left(L^2(\omega)\right)^n.$$

Moreover, let $x = (x_1, x_2, \cdots, x_n) \in \Omega$ and suppose that $\rho(x, t)$ solves the following problem

$$\begin{cases} {}_t^C D_b^\alpha \rho(x, t) = A^* \rho(x, t) & \text{in} \ \ Q, \\ \rho(\eta, t) = 0 & \text{on} \ \ \Sigma, \\ \rho(x, b) = p_{1\omega}^* z_s(x) & \text{in} \ \ \Omega, \end{cases} \tag{3.6}$$

where A^* is the adjoint operator of A and ${}_t^C D_b^\alpha$ denotes the right-sided Caputo fractional order derivative with respect to time t of order $\alpha \in (0, 1]$ and $\rho \in C(0, b; L^2(\Omega)) \cap C(0, b; H_0^1(\Omega) \cap H^2(\Omega))$ can be expressed as [27]

$$\rho(x,t)=\sum_{j=1}^{\infty}\sum_{k=1}^{r_j}E_\alpha(-\lambda_j(b-\tau)^\alpha)\,(p_{1\omega}^*z_s,\xi_{jk})\,\xi_{jk}(x), \tag{3.7}$$

simply because

$$\begin{aligned}{}_{(b-t)}{}^{C}D_b^\alpha\rho(x,b-t) &= \tfrac{-1}{\Gamma(1-\alpha)}\int_{b-t}^{b}(\tau-b+t)^{-\alpha}\tfrac{\partial}{\partial\tau}\rho(x,\tau)d\tau\\ &= \tfrac{-1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\left[-\tfrac{\partial}{\partial s}\rho(x,b-s)\right]ds\\ &= -{}_0^CD_t^\alpha\rho(x,b-t).\end{aligned} \tag{3.8}$$

Multiplying both sides of (3.1) by $\frac{\partial\rho(x,t)}{\partial x_s}$ and integrating the results over Q, we have

$$\int_Q[{}_0D_t^\alpha y(x,t)]\tfrac{\partial\rho(x,t)}{\partial x_s}dtdx=-\int_Q Ay(x,t)\tfrac{\partial\rho(x,t)}{\partial x_s}dtdx+\sum_{i=1}^{p}\int_0^b\left(f_i,\tfrac{\partial\rho(\cdot,t)}{\partial x_s}\right)_{L^2(D_i)}u_i(t)dt.$$

By Lemma 2.1, one has

$$\begin{aligned}&\int_Q[{}_0D_t^\alpha y(x,t)]\tfrac{\partial\rho(x,t)}{\partial x_s}dtdx\\ &=\int_\Omega\left[{}_0I_b^{1-\alpha}y(x,b)\right]\tfrac{\partial\rho(x,b)}{\partial x_s}dx-\int_Q y(x,t){}_t^CD_b^\alpha\left[\tfrac{\partial\rho(x,t)}{\partial x_s}\right]dtdx\\ &=\int_\Omega\left[{}_0I_b^{1-\alpha}y(x,b)\right]\tfrac{\partial\rho(x,b)}{\partial x_s}dx-\int_Q y(x,t)A^*\left[\tfrac{\partial\rho(x,t)}{\partial x_s}\right]dtdx.\end{aligned}$$

Then the boundary conditions give

$$\int_\Omega\left[{}_0I_b^{1-\alpha}y(x,b)\right]\tfrac{\partial\rho(x,b)}{\partial x_s}dx=\sum_{i=1}^{p}\int_0^b\left(p_{D_i}f_i,\tfrac{\partial\rho(\cdot,t)}{\partial x_s}\right)u_i(t)dt.$$

Since $\rho\in C(0,b;H_0^1(\Omega))$, we have

$$\int_\Omega\left[{}_0I_b^{1-\alpha}y(x,b)\right]\tfrac{\partial\rho(x,b)}{\partial x_s}dx=-\int_\Omega\frac{\partial\left[{}_0I_b^{1-\alpha}y(x,b)\right]}{\partial x_s}\rho(x,b)dx.$$

Thus, for any $z=(z_1,z_2,\cdots,z_n)\in\left(L^2(\omega)\right)^n$, it follows that

$$\begin{aligned}(p_\omega\nabla Hu,z)_{(L^2(\omega))^n}&=(\nabla Hu,p_\omega^*z)_{(L^2(\Omega))^n}\\ &=\sum_{s=1}^{n}\left(\frac{\partial\left[{}_0I_b^{1-\alpha}y(x,b)\right]}{\partial x_s},p_{1\omega}^*z_s\right)\\ &=\sum_{s=1}^{n}\left(\frac{\partial\left[{}_0I_b^{1-\alpha}y(x,b)\right]}{\partial x_s},\rho(x,b)\right)\\ &=-\sum_{s=1}^{n}\sum_{i=1}^{p}\int_0^b\left(f_i,\frac{\partial\rho(\cdot,t)}{\partial x_s}\right)_{L^2(D_i)}u_i(t)dt\\ &=-\sum_{s=1}^{n}\sum_{i=1}^{p}\sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\int_0^b E_\alpha(-\lambda_j(b-t)^\alpha)u_i(t)dt\xi_{jk}^{is}z_{jks},\end{aligned}$$

where $z_{jks}=(p_{1\omega}^*z_s,\xi_{jk})$, $s=1,2,\ldots,n$, $k=1,2,\cdots,r_j$ and $j=1,2,\cdots$. By Lemma 2.2, since $u=(u_1,u_2,\cdots,u_p)$ in (3.5) is arbitrary, $E_\alpha(-\lambda_j(b-t)^\alpha)>0$ for all $t\in[0,b]$, let $z_{js}=\left(z_{j1s},z_{j2s},\ldots,z_{jr_js}\right)^T\in\mathbf{R}^{r_j}$, we see that (3.5) is equivalent to

$$\sum_{j=1}^{\infty} E_\alpha(-\lambda_j(b-t)^\alpha) \sum_{s=1}^{n} G_j^s z_{js} = \mathbf{0}_p \;\Rightarrow z = \mathbf{0}_n. \tag{3.9}$$

(i) Assume that $p \geq r = \max\{r_j\}$ and $rank\, G_j^s < r_j$ for some $j = 1, 2, \cdots$ and $s = 1, 2, \cdots, p$, there exists a nonzero element $\tilde{z} \in \left(L^2(\omega)\right)^n$ with $\tilde{z}_{js} = \left(\tilde{z}_{j1s}, \tilde{z}_{j2s}, \cdots, \tilde{z}_{jr_js}\right)^T \in \mathbf{R}^{r_j}$ such that

$$G_j^s \tilde{z}_{js} = \mathbf{0}_p. \tag{3.10}$$

Then we can find a nonzero vector $\tilde{z} \in \left(L^2(\omega)\right)^n$ satisfying

$$\sum_{j=1}^{\infty} E_\alpha(-\lambda_j(b-t)^\alpha) \sum_{s=1}^{n} G_j^s \tilde{z}_{js} = 0. \tag{3.11}$$

This means that the actuators $(D_i, f_i)_{1 \leq i \leq p}$ are not gradient ω-strategic.

(ii) However, on the contrary, if the actuators $(D_i, g_i)_{1 \leq i \leq p}$ are not gradient ω-strategic, i.e.,

$$\overline{Im(p_\omega \nabla H)} \neq \left(L^2(\omega)\right)^n,$$

we obtain that there exists a nonzero element $z \neq \mathbf{0}_n$ satisfying

$$(p_\omega \nabla H u, z)_{(L^2(\omega))^n} = 0 \text{ for all } u \in L^2(0, b; \mathbf{R}^p). \tag{3.12}$$

Then we can find a nonzero element $z_{j^*s^*} \in \mathbf{R}^{r_j}$ such that

$$\sum_{s=1}^{n} G_{j^*}^s z_{j^*s^*} = \mathbf{0}_p. \tag{3.13}$$

This allows us to complete the first conclusion of the theorem.

In particular, when $n = s = 1$, similarly, if $p \geq r = \max\{r_j\}$ and $rank\, G_j^1 < r_j$ for some $j = 1, 2, \cdots$, we can find a nonzero vector $\tilde{z} \in \left(L^2(\omega)\right)^n$ such that

$$\sum_{j=1}^{\infty} E_\alpha(-\lambda_j(b-t)^\alpha) G_j^1 \tilde{z}_{js} = \mathbf{0}_p. \tag{3.14}$$

Then the actuators $(D_i, f_i)_{1 \leq i \leq p}$ are not gradient ω-strategic.

Moreover, if the actuators $(D_i, f_i)_{1 \leq i \leq p}$ are not gradient ω-strategic, there exists a nonzero element $z \in \left(L^2(\omega)\right)^n$ satisfying

$$G_j^1 z_{j1} = \mathbf{0}_p. \tag{3.15}$$

Then if $p \geq r = \max\{r_j\}$, it is sufficient to see that

$$rank\, G_j^1 < r_j \tag{3.16}$$

for all $j = 1, 2, \cdots$. The proof is complete. □

4. An approach for regional target control

In this section, we present an approach to steer the system (1.1) from the initial gradient vector ∇y_0 to a target gradient function $d(x) \in \left(L^2(\omega)\right)^n$ in the given subregion $\omega \subseteq \Omega$ with a minimum energy control input. The method used here is based on the Hilbert uniqueness method (HUM) developed by Lions [17] and can be also used to optimize the location of actuators during the optimal control processes.

Let U_b be the closed convex set defined by

$$U_b = \{u \in L^2\left(0, b; \mathbf{R}^p\right) : p_\omega \nabla H u = d(x)\} \tag{4.1}$$

and consider the following minimization problem

$$\inf_u J(u) = \inf_u \left\{ \int_0^b \|u(t)\|^2_{\mathbf{R}^p} dt : u \in U_b \right\}. \tag{4.2}$$

Define the gradient set G as follows

$$G := \left\{ \begin{array}{l} g \in \left(L^2(\Omega)\right)^n : g = 0 \text{ in } \Omega \backslash \omega \text{ and there exists a unique} \\ \quad \tilde{g} \in H_0^1(\Omega) \text{ such that } \nabla \tilde{g} = g \end{array} \right\}.$$

For any $g \in G$, there exists a function $\tilde{g} \in H_0^1(\Omega)$ satisfying

$$\tilde{g} = \nabla^* p_\omega^* g.$$

Consider the system

$$\begin{cases} T_t^C D_b^\alpha \varphi(x,t) = A^* T\varphi(x,t) \quad \text{in } Q, \\ \varphi(\eta, b-t) = 0 \quad \text{on } \Sigma, \\ \lim\limits_{t \to 0^+} \varphi(x, b-t) = \nabla^* p_\omega^* g(x) \quad \text{in } \Omega, \end{cases} \tag{4.3}$$

where T is a reflection operator on interval $[0, b]$:

$$Tf(t) := f(b-t). \tag{4.4}$$

For more properties on the operator T, please see the monograph [14]. Moreover, by (3.8), we see that the system (4.3) admits a unique solution $\varphi \in L^2(0, b; H_0^1(\Omega)) \cap C([0, b] \times \Omega)$ given by [27]

$$\varphi(x,t) = \sum_{j=1}^{\infty} E_\alpha(-\lambda_j (b-t)^\alpha) \left(\xi_j, \nabla^* p_\omega^* g\right) \xi_j(x). \tag{4.5}$$

Consider the following semi-norm on G

$$g \in G \to \|g\|_G^2 = \int_0^b \|B^* \varphi(\cdot, t)\|^2 dt \tag{4.6}$$

and we obtain the following preliminary results.

Lemma 4.1. *If the system (1.1) is regionally approximately gradient controllable in ω at time b, then (4.6) defines a norm on G.*

Proof. If the system (1.1) is regionally approximately gradient controllable in ω at time b, by Definition 2.1, one has

$$H^*\nabla^* p_\omega^* g = 0 \Rightarrow g = \mathbf{0}_n. \tag{4.7}$$

Further, for any $g \in G$, since

$$\|g\|_G = 0 \Leftrightarrow B^*\varphi(x,t) = 0, \tag{4.8}$$

it then follows that (4.6) defines a norm on G and the proof is complete. □

In addition, consider the following system

$$\begin{cases} {}_0D_t^\alpha \psi(x,t) + A\psi(x,t) = BB^*\varphi(x,t) \quad \text{in } Q, \\ \psi(\eta,t) = 0 \quad \text{on } \Sigma, \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha}\psi(x,t) = 0 \quad \text{in } \Omega, \end{cases} \tag{4.9}$$

which is controlled by the solution of the system (4.3). Let $\Lambda : G \to G^*$ be

$$\Lambda g = p_\omega \nabla {}_0I_b^{1-\alpha}\psi(\cdot,b). \tag{4.10}$$

Suppose that $\psi_0(x,t)$ satisfies

$$\begin{cases} {}_0D_t^\alpha \psi_0(x,t) + A\psi_0(x,t) = 0 \ \text{ in } Q, \\ \psi_0(\eta,t) = 0 \quad \text{on } \Sigma, \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha}\psi_0(x,t) = y_0(x) \quad \text{in } \Omega. \end{cases} \tag{4.11}$$

Then the regional gradient controllability problem is equivalent to solving the equation

$$\Lambda g := d(x) - p_\omega \nabla {}_0I_b^{1-\alpha}\psi_0(x,b), \quad d \in \left(L^2(\omega)\right)^n. \tag{4.12}$$

Theorem 4.1. *For any $d \in \left(L^2(\omega)\right)^n$, if (1.1) is regionally approximately gradient controllable in ω at time b, then (4.12) admits a unique solution $g \in G$, the control*

$$u^*(t) = B^*\varphi(x,t) \tag{4.13}$$

steers the gradient of the system (1.1) to $d(x)$ at time b and solves the minimum energy problem (4.2).

Proof. If the system (1.1) is regionally approximately gradient controllable in ω at time b, it follows from Lemma 4.1 that $\|\cdot\|_G$ defines a norm on G. Let us denote the completion of G with respect to the norm $\|\cdot\|_G$ again by G. Then we first show that (4.12) admits a unique solution in G with the help of the Theorem 1.1 in [16].

Since U_b is convex, by the Theorem 1.1 in [16], to obtain the existence of the unique solution for the problem (4.12), we only need to show that Λ is coercive from G to G^*, i.e., there exists a constant $\mu > 0$ such that

$$(\Lambda g, g)_{(L^2(\Omega))^n} \geq \mu \|g\|_G^2, \quad \forall g \in G. \tag{4.14}$$

Indeed, for any $g \in G$, it follows from the definition of operator Λ in (4.10) that

$$(g, \Lambda g) = \left(g, p_\omega \nabla_0 I_b^{1-\alpha} \psi(\cdot, b)\right) = \int_0^b (\varphi(\cdot, s), BB^* \varphi(\cdot, s)) ds = \|g\|_G^2.$$

Hence, $\Lambda : G \to G^*$ is one to one and (4.12) admits a unique solution in G.

Suppose that g^* is the unique solution of (4.12), by the argument above, it is easy to see that the system (1.1) can be steered from the initial gradient vector ∇y_0 to any gradient vector $d(x) \in \left(L^2(\omega)^n\right)$ at time b by the control

$$u^*(t) = B^* \varphi(x, t) = B^* \sum_{j=1}^{\infty} E_\alpha(-\lambda_j (b-t)^\alpha) \left(\xi_j, \nabla^* p_\omega^* g^*\right) \xi_j(x),$$

i.e.,

$$p_\omega \nabla_0 I_b^{1-\alpha} y(b, u^*) = d(x).$$

Finally, we present that u^* solves the minimum energy problem (4.2). For any $u \in L^2(0, b, \mathbf{R}^p)$ with

$$p_\omega \nabla_0 I_b^{1-\alpha} y(b, u) = d(x),$$

we have

$$p_\omega \nabla_0 I_b^{1-\alpha} \left[y(b, u) - y(b, u^*)\right] = \mathbf{0}_n \tag{4.15}$$

and for any $g \in G$,

$$\begin{aligned} 0 &= \left(g, p_\omega \nabla_0 I_b^{1-\alpha} \left(y(b, u) - y(b, u^*)\right)\right)_{(L^2(\Omega))^n} \\ &= \left(\nabla^* p_\omega^* g, {}_0 I_b^{1-\alpha} \left(y(b, u) - y(b, u^*)\right)\right) \\ &= \int_0^b \sum_{j=1}^{\infty} E_\alpha(-\lambda_j (b-s)^\alpha) \left(\nabla^* p_\omega^* g, \xi_j\right) \left(\xi_j, B[u(s) - u^*(s)]\right) ds \\ &= \int_0^b (B^* \varphi(\cdot, s), [u(s) - u^*(s)]) ds. \end{aligned} \tag{4.16}$$

It then follows that

$$J'(u^*)(u - u^*) = 2 \int_0^b (u^*(s), u(s) - u^*(s)) ds = 2 \int_0^b (B^* \varphi(\cdot, s), u(s) - u^*(s)) ds = 0.$$

By Theorem 1.3 in [16], we conclude u^* is the solution of the minimum energy problem (4.2) and the proof is complete. □

5. Applications

In this section, we focus on the following system in $\mathbf{R}^2$

$$\begin{cases} {}_0 D_t^\alpha y(x, t) = \triangle y(x, t) + Bu(t) & \text{in } \Omega \times [0, b], \\ y(\eta, t) = 0 & \text{on } \partial\Omega \times [0, b], \\ \lim\limits_{t \to 0^+} {}_0 I_t^{1-\alpha} y(x, t) = 0 & \text{in } \Omega, \end{cases} \tag{5.1}$$

where $\triangle$ is the elliptic operator defined by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}. \tag{5.2}$$

Our objective here is to find the minimum energy control to derive the system (5.1) from the initial gradient vector ∇y_0 to any gradient vector in $\left(L^2(\omega)\right)^2$. The actuators may be zone actuators, pointwise actuators or filament actuators and it is supposed that $\Omega = [0,1] \times [0,1]$, $\omega \subseteq \Omega$.

5.1. Zone actuators

Consider the system (5.1) with the following controller

$$Bu(t) = \sum_{i=1}^{p} p_{D_i} f_i(x) u_i(t), \tag{5.3}$$

where $D_i \subseteq \Omega$, $f_i \in L^2(\Omega)$, $i = 1, 2, \cdots, p$ and $B \in \mathcal{L}\left(\mathbf{R}^p, L^2(\Omega)\right)$ is bounded.

Let $A = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$. Then the eigenvalues are $\lambda_{mn} = (m^2 + n^2)\pi^2$, corresponding eigenfunctions are $\xi_{mn}(x_1, x_2) = 2\sin(m\pi x_1)\sin(n\pi x_2)$ and the semigroup is

$$\Phi(t)y(x) = \sum_{m,n=1}^{\infty} \exp(-\lambda_{mn} t)(y, \xi_{mn})\xi_{mn}(x), \;\; y \in L^2(\Omega).$$

Then the multiplicity of the eigenvalues is one and we have

$$H^* g = \sum_{m,n=1}^{\infty} \sum_{i=1}^{p} E_\alpha(-\lambda_{mn}(b-t)^\alpha)\,(\xi_{mn}, g)\,(\xi_{mn}, f_i)_{L^2(D_i)}\,.$$

By Theorem 3.1, we obtain

Proposition 5.1. *There exists a subregion $\omega \subseteq \Omega$ such that the actuators $(D_i, p_{D_i} f_i)_{1 \le i \le p}$ are gradient ω-strategic at time b if and only if*

$$\left\{ \begin{array}{l} m \int_{D_i} f_i(x_1, x_2)\cos(m\pi x_1)\sin(n\pi x_2)dx_1 dx_2 \;\; z_{j1} \\ +n \int_{D_i} f_i(x_1, x_2)\sin(m\pi x_1)\cos(n\pi x_2)dx_1 dx_2 \;\; z_{j2} = 0 \end{array} \right\} \Rightarrow z = (z_1, z_2) = (0, 0) \tag{5.4}$$

for all $i = 1, 2, \cdots, p$, $m, n = 1, 2, \cdots$, where $z_{js} = (p^*_{1\omega} z_s, \xi_j)$, $s = 1, 2$.

Proof. According to the argument above, we have $r_j = 1$, $n = 2$. It then follows that

$$G_j^1 = 2m\pi \begin{bmatrix} \int_{D_1} f_1(x_1, x_2)\cos(m\pi x_1)\sin(n\pi x_2)dx_1 dx_2 \\ \int_{D_2} f_2(x_1, x_2)\cos(m\pi x_1)\sin(n\pi x_2)dx_1 dx_2 \\ \vdots \\ \int_{D_p} f_p(x_1, x_2)\cos(m\pi x_1)\sin(n\pi x_2)dx_1 dx_2 \end{bmatrix}_{p\times 1}, \tag{5.5}$$

$$G_j^2 = 2n\pi \begin{bmatrix} \int_{D_1} f_1(x_1, x_2)\sin(m\pi x_1)\cos(n\pi x_2)dx_1 dx_2 \\ \int_{D_2} f_2(x_1, x_2)\sin(m\pi x_1)\cos(n\pi x_2)dx_1 dx_2 \\ \vdots \\ \int_{D_p} f_p(x_1, x_2)\sin(m\pi x_1)\cos(n\pi x_2)dx_1 dx_2 \end{bmatrix}_{p\times 1} \tag{5.6}$$

for any $j = 1, 2, \cdots$. By Theorem 3.1, the actuators $(D_i, p_{D_i} f_i)_{1\le i\le p}$ are gradient ω_2-strategic at time b if and only if

$$\sum_{s=1}^{2} G_j^s z_{js} = \mathbf{0}_p \ \Rightarrow z = (z_1, z_2) = (0, 0) \text{ in } \left(L^2(\omega)\right)^2 \text{ for all } m, n = 1, 2, \cdots, \tag{5.7}$$

i.e., (5.4) holds and the proof is complete. □

Next, suppose that the system (5.1) with the control input (5.3) is regionally approximately gradient controllable on ω at time b, by Lemma 4.1, then

$$\|g\|_{G_2}^2 = \int_0^b \left\| \sum_{i=1}^{p} (f_i, \varphi(\cdot, t))_{L^2(D_i)} \right\|^2 dt \tag{5.8}$$

defines a norm on the space

$$G_2 = \left\{ g \in L^2(\Omega) : g = 0 \text{ in } \Omega\backslash\omega \text{ and } \exists\ \tilde{g} \in H_0^1(\Omega) \text{ such that } \nabla\tilde{g} = g \right\}, \tag{5.9}$$

where $\varphi(x, t)$ is the solution of the following system (5.14). It then follows from Theorem 4.1 that if the system (5.1) with the control input (5.3) is regionally approximately gradient controllable on ω at time b,

$$\Lambda g(x) = p_\omega \nabla {}_0I_b^{1-\alpha} \psi(x, b) = d(x), \quad d \in \left(L^2(\omega)\right)^2, \tag{5.10}$$

admits a unique solution g^* in G_2 and the minimum energy control is given by

$$u^*(t) = \sum_{i=1}^{p} (f_i, \varphi(\cdot, t))_{L^2(D_i)},$$

where $\psi(x, t)$ solves the following problem

$$\begin{cases} {}_0D_t^\alpha \psi(x, t) = \triangle\psi(x, t) + \sum\limits_{i=1}^{p} p_{D_i} f_i(x) \left(f_i, \varphi(\cdot, t)\right)_{L^2(D_i)}, & (x, t) \in \Omega \times [0, b], \\ \varphi(\eta, b - t) = 0, \quad (\eta, t) \in \partial\Omega \times [0, b], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} \psi(x, t) = 0, \quad x \in \Omega. \end{cases}$$

5.2. Pointwise actuators

In this part, we consider the problem (5.1) excited by p-pointwise actuators $(\sigma_i, \delta_{\sigma_i})_{1\le i\le p}$ and

$$Bu(t) = \sum_{i=1}^{p} \delta_{\sigma_i} u_i(t), \tag{5.11}$$

where $\sigma_i \in \Omega$, $\delta_{\sigma_i} = \delta(\cdot - \sigma_i)$ is a generalized function in ω at time b that is zero everywhere except at σ_i, $i = 1, 2, \cdots, p$. Then the operator B is a densely defined operator and B^* exists. Hence, the assumption (A_1) holds.

Since $|\xi_{mn}| \le 2$ for $x \in [0, 1] \times [0, 1]$, $E_{\alpha,\alpha}(\lambda_{mn} t^\alpha)$ is continuous [23],

$$|E_{\alpha,\alpha}(\lambda_{mn} t^\alpha)| \le \frac{C}{1 + |\lambda_{mn}| t^\alpha} \in L^2(0, b) \ (C > 0)$$

and

$$H^*g = \sum_{m,n=1}^{\infty} \sum_{i=1}^{p} E_\alpha(-\lambda_{mn}(b-t)^\alpha)\,(\xi_{mn}, g)\,\xi_{mn}(\sigma_i).$$

Then the assumption (A_2) is satisfied. From the discussion in Example 2.1, the system (5.1) with the control input (5.11) is not approximately gradient controllable on Ω. However, by Theorem 3.1, similar to the argument in Proposition 5.1, we obtain the following results.

Proposition 5.2. *There exists a subregion $\omega \subseteq \Omega$ such that the actuators $(\sigma_i, \delta_{\sigma_i})_{1\le i\le p}$ are gradient ω-strategic at time b if and only if both*

$$\left\{\begin{array}{l} m\cos(m\pi\sigma_{i1})\sin(n\pi\sigma_{i2})z_{j1} \\ \quad + n\sin(m\pi\sigma_{i1})\cos(n\pi\sigma_{i2})z_{j2} = 0 \end{array}\right\} \Rightarrow z = (z_1, z_2) = (0, 0) \tag{5.12}$$

for $i = 1, 2, \cdots, p$, $m, n = 1, 2, \cdots$, $\sigma_i = (\sigma_{i1}, \sigma_{i2}) \in \Omega$ and $z_{js} = (p^*_{1\omega} z_s, \xi_j)$, $s = 1, 2$.

Next, suppose that the example (5.1) with the control input (5.11) is regionally approximately gradient controllable on ω at time b, by Lemma 4.1, we have

$$\|g\|^2_{G_2} = \int_0^b \left\| \sum_{i=1}^{p} \varphi(\sigma_i, t) \right\|^2 dt \tag{5.13}$$

defines a norm on the space G_2, where $\varphi(x,t)$ is the solution of the following system

$$\begin{cases} T_t^C D_b^\alpha \varphi(x,t) = \triangle T\varphi(x,t), \quad (x,t) \in \Omega\times[0,b], \\ \varphi(\eta, b-t) = 0, \quad (\eta,t) \in \partial\Omega\times[0,b], \\ \lim\limits_{t\to 0^+} \varphi(x, b-t) = \nabla^* p^*_\omega g(x), \quad x \in \Omega. \end{cases} \tag{5.14}$$

Then the regional gradient controllability problem is equivalent to solving the equation

$$\Lambda g(x) = p_\omega \nabla_0 I_b^{1-\alpha}\psi(x,b) = d(x), \quad d \in \left(L^2(\omega)\right)^2, \tag{5.15}$$

where $\psi(x,t)$ is the solution of the problem

$$\begin{cases} {}_0D_t^\alpha \psi(x,t) = \triangle\psi(x,t) + \sum\limits_{i=1}^{p} \varphi(\sigma_i, t), (x,t) \in \Omega\times[0,b], \\ \varphi(\eta, b-t) = 0, \quad (\eta,t) \in \partial\Omega\times[0,b], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha}\psi(x,t) = 0, \quad x \in \Omega \end{cases}$$

controlled by the solution of the system (5.14). It follows from Theorem 4.1 that (5.15) admits a unique solution g^* in G_2 and the minimum energy control can be given by

$$u^*(t) = \sum_{i=1}^{p} \varphi(\sigma_i, t). \tag{5.16}$$

5.3. Filament actuators

Consider the case where the actuators $(F_i, \delta_{F_i})_{1\le i\le p}$ are located on the curve $F_i = Im(\mu_i)$ with $\mu_i \in C^1(0,1)$ and the controller is given by

$$Bu(t) = \sum_{i=1}^{p} \delta_{F_i} u_i(t). \tag{5.17}$$

Suppose that $F_i = [\tau_{i1}, \tau_{i2}] \times \{\sigma_i\} \subseteq \Omega$, $i = 1, 2, \cdots, p$, by Theorem 3.1, similar to the argument in Proposition 5.1, we see that

Proposition 5.3. *There exists a subregion $\omega \subseteq \Omega$ such that the actuators $(F_i, \delta_{F_i})_{1\le i\le p}$ are gradient ω-strategic at time b if and only if*

$$\left\{ \begin{array}{l} m\sin(n\pi\sigma_i)\int_{\tau_{i1}}^{\tau_{i2}} \delta_{F_i}(x_1,\sigma_i)\cos(m\pi x_1)dx_1 \ z_{j1} \\ \quad + n\cos(n\pi\sigma_i)\int_{\tau_{i1}}^{\tau_{i2}} \delta_{F_i}(x_1,\sigma_i)\sin(m\pi x_1)dx_1 \ z_{j2} = 0 \end{array} \right\} \Rightarrow z = (z_1, z_2) = (0,0) \tag{5.18}$$

for all $i = 1, 2, \cdots, p$, $m, n = 1, 2, \cdots$ and $z_{js} = (p_{1\omega}^* z_s, \xi_j)$, $s = 1, 2$.

Let $\varphi(x,t)$ be a solution of the system (5.14). It follows from Lemma 4.1 that

$$\|g\|_{G_2}^2 = \int_0^b \left\| \sum_{i=1}^{p} (\delta_{F_i}, \varphi(\cdot,t)) \right\|^2 dt \tag{5.19}$$

defines a norm on G_2 provided (5.1) with the control input (5.17) is regionally approximately gradient controllable on ω at time b.

Moreover, by Theorem 4.1, if the system (5.1) with the control input (5.17) is regionally approximately gradient controllable on ω at time b,

$$\Lambda g(x) = p_\omega \nabla_0 I_b^{1-\alpha}\psi(x,b) = d(x), \quad d \in \left(L^2(\omega)\right)^2, \tag{5.20}$$

admits a unique solution g^* in G_2 and the minimum energy control is given by

$$u^*(t) = \sum_{i=1}^{p} (\delta_{F_i}, \varphi(\cdot,t)),$$

where $\psi(x,t)$ solves the following problem

$$\begin{cases} {}_0D_t^\alpha \psi(x,t) = \triangle\psi(x,t) + \sum\limits_{i=1}^{p} \delta_{F_i}(x)\left(\delta_{F_i}, \varphi(\cdot,t)\right), (x,t) \in \Omega\times[0,b], \\ \varphi(\eta, b-t) = 0, \qquad (\eta,t) \in \partial\Omega\times[0,b], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha}\psi(x,t) = 0, \quad x\in\Omega. \end{cases}$$

6. Conclusion

This paper is concerned with the regional gradient controllability for the Riemann–Liouville time fractional order diffusion system of order $\alpha \in (0,1]$, which is motivated by many real world applications where the objective is to explore the minimum energy control to steer the system under consideration from the initial gradient vector ∇y_0 to any gradient vector in an interested subregion of the whole domain. We hope that the results here could provide some insight into the control theory analysis of fractional order system. The results presented here can also be extended to complex fractional order distributed parameter systems and various open questions are still under consideration. For instance, the problem of constrained regional gradient control of fractional order distributed parameter systems with more complicated regional sensing and actuation configurations are of great interest. For more information on the potential topics related to fractional distributed parameter systems, we refer the readers to [7] and the references therein.

Acknowledgments

The authors would like to thank Chinese Universities Scientific Fund (No. CUSF-DH-D-2014061) and the Natural Science Foundation of Shanghai (No. 15ZR1400800).

References

[1] L. Afifi, A. Chafia, A.E. Jai, Regionally efficient and strategic actuators, Internat. J. Systems Sci. 33 (1) (2002) 1–12.
[2] Á. Cartea, D. del Castillo-Negrete, Fluid limit of the continuous-time random walk with general Lévy jump distribution functions, Phys. Rev. E 76 (4) (2007) 1–8.
[3] R. Courant, D. Hilbert, Methods of Mathematical Physics, CUP Archive, 1966.
[4] B. Dacorogna, Direct Methods in the Calculus of Variations, second edition, Springer Science & Business Media, 2007.
[5] A. El Jai, Distributed systems analysis via sensors and actuators, Sens. Actuators A Phys. 29 (1) (1991) 1–11.
[6] A. El Jai, A.J. Pritchard, Sensors and Controls in the Analysis of Distributed Systems, Halsted Press, 1988.
[7] F. Ge, Y. Chen, C. Kou, Cyber-physical systems as general distributed parameter systems: three types of fractional order models and emerging research opportunities, IEEE/CAA J. Autom. Sin. 2 (4) (2015) 353–357.
[8] F. Ge, Y. Chen, C. Kou, Regional controllability of anomalous diffusion generated by the time fractional diffusion equations, in: ASME IDETC/CIE 2015, Boston, Aug. 2–5, 2015, DETC2015-46697, in press. See also: arXiv preprint, arXiv: 1508.00047.
[9] F. Ge, Y. Chen, C. Kou, Actuator characterizations to achieve approximate controllability for a class of fractional subdiffusion equations, Internat. J. Control (2016), http://dx.doi.org/10.1080/00207179.2016.1163619, in press.
[10] R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogosin, Mittag-Leffler Functions, Related Topics and Applications, Springer, 2014.
[11] G. Gradenigo, A. Sarracino, D. Villamaina, A. Vulpiani, Einstein relation in systems with anomalous diffusion, Acta Phys. Polon. B 44 (5) (2013) 899–912.
[12] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, 1952.
[13] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science Limited, 2006.
[14] M. Klimek, On Solutions of Linear Fractional Differential Equations of a Variational Type, Publishing Office of Czestochowa University of Technology, 2009.
[15] M. Kurula, H. Zwart, The Duality Between the Gradient and Divergence Operators on Bounded Lipschitz Domains, Department of Applied Mathematics, University of Twente, October 2012, http://www.math.utwente.nl/publications.
[16] J.L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, 1971.
[17] J.L. Lions, Exact controllability, stabilization and perturbations for distributed systems, SIAM Rev. 30 (1) (1988) 1–68.
[18] B.B. Mandelbrot, The Fractal Geometry of Nature, Macmillan, 1983.
[19] M.M. Meerschaert, E. Scalas, Coupled continuous time random walks in finance, Phys. A, Stat. Mech. Appl. 370 (1) (2006) 114–118.
[20] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep. 339 (1) (2000) 1–77.
[21] E.W. Montroll, G.H. Weiss, Random walks on lattices. II, J. Math. Phys. 6 (2) (1965) 167–181.
[22] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, vol. 44, Springer Science & Business Media, 2012.
[23] I. Podlubny, Fractional Differential Equations, Academic Press, 1999.
[24] I. Podlubny, Y. Chen, Adjoint fractional differential expressions and operators, in: ASME 2007 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, American Society of Mechanical Engineers, 2007, pp. 1385–1390.
[25] A. Pritchard, A. Wirth, Unbounded control and observation systems and their duality, SIAM J. Control Optim. 16 (4) (1978) 535–545.
[26] M. Renardy, R.C. Rogers, An Introduction to Partial Differential Equations, Springer Science & Business Media, 2006.
[27] K. Sakamoto, M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems, J. Math. Anal. Appl. 382 (1) (2011) 426–447.
[28] L. Sario, G. Weill, Normal operators and uniformly elliptic self-adjoint partial differential equations, Trans. Amer. Math. Soc. (1965) 225–235.

[29] A. Sharifi-Viand, M. Mahjani, M. Jafarian, Investigation of anomalous diffusion and multifractal dimensions in polypyrrole film, J. Electroanal. Chem. 671 (2012) 51–57.

[30] P.J. Torvik, R.L. Bagley, On the appearance of the fractional derivative in the behavior of real materials, J. Appl. Mech. 51 (2) (1984) 294–298.

[31] H. Weinberger, Symmetrization in uniformly elliptic problems, in: Studies in Math. Anal., Stanford Univ. Press, 1962, pp. 424–428.

[32] E. Zerrik, A. Boutoulout, A. Kamal, Regional gradient controllability of parabolic systems, Int. J. Appl. Math. Comput. Sci. 9 (4) (1999) 767–787.

[33] E. Zerrik, A. Kamal, A. Boutoulout, Regional gradient controllability and actuators, Internat. J. Systems Sci. 33 (4) (2002) 239–246.