Is Our Universe Expanding Dynamics Fractional Order?

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Nov. 6, 2012. Tuesday 11:00-12:00
Castle Research Facility Room #22
Motivation

Supernovae dataset

Nobel Price in Physics 2006: G. Smoot & J. Mather

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Is Our Universe Expanding Dynamics Fractional Order?

◆ Before the time of Einstein:
  Static (Cosmological Constant)

◆ Since 1929 (Edwin Hubble):
  Expanding

◆ Since 1998 (Riess et al. & Perlmutter et al.):
  Accelerating (Dark Energy)

◆ Today:
  Accelerating Law---Fractional Calculus
How to Measure Our Expanding Universe?

Models of the Expanding Universe

- Densely, rapidly decelerating universe
- Sparse, slowly decelerating universe
- Decelerating, then accelerating universe

Figure credit: Dr. Adam G. Riess

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Type Ia Supernovae (SNIa)--- Standard Candle

Distance measurement:
The relative age of the Universe at the time of the supernova explosion

Redshift measurement:
The growth factor of the Universe at the time a supernova exploded

Hubble Diagram

Figure credit: Dr. Adam G. Riess
This relationship between distance modulus and redshift can be considered as a purely kinematic record of the universe’s expansion history.

That is to say, the relative positions of galaxies can tell us whether the universe was ever accelerating or decelerating, regardless of its cause.
Findings:
Hubble showed that galaxies recede from us in all directions and more distant ones recede more rapidly in proportion to their distance.

The Universe is expanding!

\[ H_0 = 72 \]

13.7 billion years

(E.P. Hubble, PNAS 15 (1929) 168-173.)

Hubble’s Law:
\[ c \ z = \nu = H_0 \ d \]


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Slide credit: [Supernova Cosmology Project](https://supernova.lbl.gov/)

Cluster Search (SCP)
Amanullah et al. (2010) (SCP)
Riess et al. (2007)
Tonry et al. (2003)

Miknaitis et al. (2007)
Astier et al. (2006)
Knop et al. (2003) (SCP)
Amanullah et al. (2008) (SCP)
Barris et al. (2004)
Perlmutter et al. (1999) (SCP)
Riess et al. (1998) + HZT
Holtzman et al. (2009)

Contreras et al. (2010)
Hicken et al. (2009)
Kowalski et al. (2008) (SCP)
Jha et al. (2006)
Riess et al. (1999)
Kris ciunas et al. (2005)
Hamuy et al. (1996)
“Fractional Order Thinking”

IS OUR UNIVERSE EXPANDING DYNAMICS FRACTIONAL ORDER?

WHAT IS THE BENEFIT BY USING FRACTIONAL CALCULUS?
\[
\cd_{0}^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \int_{0}^{t} \frac{f^{(n)}(\tau)}{(t - \tau)^{\alpha-n+1}} d\tau, \quad (n - 1 < \alpha \leq n, \ n \in \mathbb{N})
\]

\[
\mathcal{L}\left\{\cd_{0}^{\alpha} f(t) ; s \right\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)
\]
Mittag-Leffer function:

\[ E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k \cdot \alpha + \beta)}, \quad \Re(\alpha) > 0, \ Re(\beta) > 0 \]

\[ \mathcal{L}\{t^{\beta-1} E_{\alpha,\beta}(\pm \lambda t^\alpha); s\} = \frac{s^{\alpha - \beta}}{s^{\alpha} \mp \lambda} \]

Special cases:

\[ E_{0,1}(t) = \frac{1}{1 - t}, \quad E_{1,1}(t) = \exp(t), \quad E_{1,2}(t) = \frac{\exp(t) - 1}{t} \]
Data fitting → Accelerating law → Mittag-Leffler function
Approach: \[ f(t) = K t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad (1 < \alpha < 2) \]

\[ f(t) = \int_0^t y(\tau) g(t - \tau) d\tau \]

Where \[ g(t) = \frac{K}{y_1} t^{\alpha-2} E_{1,\alpha-1} \left( -\frac{y_2}{y_1} t \right) \] and

\[ ^CD_0^\alpha y(t) + \lambda y(t) = 0, \quad y(0) = y_1, \quad y'(0) = y_2 \]
Dataset: \([z(k), \mu(k)]\)

http://supernova.lbl.gov/Union/figures/SCPUnion2.1_mu_vs_z.txt

Fitting function: \([z(k), f(k)]\)

\[
f(t) = K t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad (1 < \alpha < 2)
\]

Aim: \(J = \min \sqrt{\frac{\sum_{k=1}^{580} (f(k) - \mu(k))^2}{580}}\)

580: Number of the type Ia supernovae up to date

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Matlab Codes

```matlab
clear all;clc;
options=optimset('TolX',1e-15,'TolFun',1e-15);
z=importdata('z.mat');
mu=importdata('mu.mat');
[x,FVAL,EXITFLAG]=fminsearch(@mf,mlf_scpx,mu,z,[1,2,1.1],options);
K=x(1);lambda=x(2);alpha=x(3);
y=mlf(alpha,alpha,-lambda*z.^alpha);
y=K*y.*z.^(alpha-1);
figure;
scatter(z,mu)
hold on
plot(z,y,'k','LineWidth',2);
grid on
xlabel('Redshift z');ylabel('Distance Modulus \mu');

function [J]=mlf_scpx(x,y0,z)
K=x(1);lambda=x(2);alpha=x(3);
N=length(y0);
y=mlf(alpha,alpha,-lambda*z.^alpha);
y=K*y.*z.^(alpha-1);
J=sqrt(((y-y0)*(y-y0))'/N);
```

http://www.mathworks.com/matlabcentral/fileexchange/8738
Results

\[ K = 42.7638 \]
\[ \alpha = 1.0616 \]
\[ \lambda = 9.5671 \times 10^{-4} \]

\[ f(t) = 42.7638 t^{0.0616} E_{1.0616,1.0616} (-9.5671 \times 10^{-4} t^{1.0616}) \]

\[ f(t) = \int_0^t y(\tau) g(t - \tau) d\tau \]

\[ g(t) = \frac{42.7638}{y_1 t^{0.9384}} E_{1,0.0616} \left(-\frac{y_2}{y_1} t\right) \]

\[ CD_0^{1.0616} y(t) + 9.5671 \times 10^{-4} y(t) = 0, \quad y(0) = y_1, \quad y'(0) = y_2 \]

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### M-L function VS Exp functions

<table>
<thead>
<tr>
<th>Fitting functions</th>
<th>( J )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1(t) = K t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) )</td>
<td>0.2649</td>
<td>3</td>
</tr>
<tr>
<td>( f_2(t) = \sum_{i=1}^{n} K_i \exp(-\lambda_i t) )</td>
<td>n=1 1.5874</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>n=2 1.0680</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>n=3 0.5053</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>n=4 0.9241</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>n=5 2.7227</td>
<td>10</td>
</tr>
<tr>
<td>( f_3(t) = \sum_{i=1}^{n} K_i \exp(-\lambda_i t^\alpha) )</td>
<td>n=1 0.2687</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>n=2 0.2671</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>n=3 0.2691</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>n=4 0.2672</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>n=5 0.2674</td>
<td>11</td>
</tr>
</tbody>
</table>

\( J \): Square Error, \( N \): Number of Parameters
Benefit by Using Fractional calculus

1. The evolution of expanding dynamics of our universe obeys a Mittag-Leffler function, meaning that the accelerating law can be described by some fractional order differential equation, whose solution is the Mittag-Leffler function.
2. It can help the astronomers to discover new Type Ia supernovae with high redshift according to the proposed Mittag-Leffler function.
3. The fitting model is simpler than other models.

\[ \mu_0 = m - M = 5 \log d_L + 25 \]

\[ d_L = c(1 + z) \int_0^z \frac{du}{H(u)} \]

\[ = c(1 + z) H_0^{-1} \int_0^z \exp \left\{ - \int_0^u [1 + q(u)]d \ln (1 + u) \right\} du, \]

\[ H(z) = \frac{\dot{a}}{a}, \quad q(z) \equiv \frac{-\ddot{a}/a}{H^2(z)} = \frac{dH^{-1}(z)}{dt} - 1. \]

\[ a(t) = a_0 \left\{ 1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2(t - t_0)^2 \right. \]

\[ + \frac{1}{3!} j_0 H_0^3(t - t_0)^3 + O[(t - t_0)^4] \right\}, \]

\[ j(t) = + \left( \frac{\ddot{a}}{a} \right) \left( \frac{\dot{a}}{a} \right)^{-3}. \]
Discussion

\[ f(t) = K t^{\alpha - 1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad (1 < \alpha < 2) \]

Where \( g(t) = \frac{K}{y_1} t^{\alpha - 2} E_{1,\alpha - 1} \left( -\frac{y_2}{y_1} t \right) \) and

\[ {}^C D_0^\alpha y(t) + \lambda y(t) = 0, \quad y(0) = y_1, \quad y'(0) = y_2 \]
Discussion

• Variable order:

\[ \alpha \rightarrow \alpha(t) \]

• Distributed order:

\[ D_{(\varphi)} f(t) = \int_{\beta_1}^{\beta_2} \varphi(\alpha) D^\alpha f(t) d\alpha \]
Thank you for your attention!

All Questions are welcome!