Research on the Higher-order Logic Formalization of Fractance Element

Shanshan Li, Chunna Zhao, *Member, IEEE,* Yong Guan, *Member, IEEE,* Zhiping Shi, *Member, IEEE CAA,* Xiaojuan Li, *Member, IEEE,* Rui Wang, *Member, IEEE,* and Qianying Zhang

***Abstract*—Fractance element reflects the fractional order behavior of circuits, which can show the characteristics of the actual circuits. Higher-order logic theorem proving is based on the rigorous and correct mathematical theories. It becomes more and more important in the verifications of high-reliability systems. Fractance element is formalized using the proof of higher-order logic theorem in this paper. Firstly, the formalized model of fractional calculus which is based on Caputo definition is established in higher-order logic theorem proof tool. Then some properties of fractional calculus are proved, including the zero order property, the fractional differential of a constant and the consistency of fractional calculus and integer order calculus. Finally, fractance element and fractional differential circuit constituted by fractance element are formally analyzed. These formalizations demonstrate the effectiveness of the formal method in the analysis of fractance element.**

***Index Terms*—Fractional Calculus, Caputo Definition, Theorem Proving, Fractance Element, Fractional Differential Circuit.**

I. INTRODUCTION

FRACTANCE element is a component with fractional order impedance. It can accomplish the function of fractional calculus for signal. Fractance element is different from the impedance, capacitive reactance or inductive reactance. It can show the characteristics between capacitance and inductance [1]. Fractance circuit refers to the circuit which includes fractance elements. Components in the circuits are often considered to be a resistance, capacitance, or inductance which is with integer order. However, due to the materials or other reasons, the components in actual circuits do not show these desirable characteristics. Actually, they present the characteristics between these ideal characteristics. Ignoring these facts will lead to inaccurate modeling. In addition, accuracy problems will emerge if we adjust the circuits according to the misconception that the components present ideal characteristics. Fractional calculus is the theoretical basis of fractance element and fractance circuit. It can be used to effectively describe the dynamic process of some systems which cannot be accurately described by integer order calculus [2]. Now, fractional calculus is widely used in hyperchaotic system [3], viscoelastic system [4], anomalous diffusion, fluid dynamics [5], image processing [6], signal processing, seismic analysis, control of robot [7], electric power transmission line [8], fuzzy control [9] and other fields. Studies have shown that the models using fractional calculus can better and more accurately describe the characteristics of actual systems [10].

The current phase of research on fractance element and fractance circuit is concerned with their realization. For instance, the realizations of fractance element and fractance circuit are discussed in references $[11-13]$. In reference [11], the fractional order operator is realized by using the finite inertia and the cascade of proportional differential circuit. Reference[12] presents an implementation of variable order analog circuit by using operational transconductance amplifier. Besides, the realization of fractional analog circuit by using the method of binomial expansion is given in reference[13]. A wide variety of implementation schemes have been proposed and these schemes have also been obtained in some applications. However, few studies have focused on the analysis of the fractance element and fractance circuit. For the analysis of fractance element and fractance circuit, the traditional methods include paper-and-pencil based proofs, analog simulation and computer algebra system. The results of these methods cannot achieve 100% rate of precision because of the cumbersome process, approximation errors, difficulty in building environment for application of these methods and that the algorithms for symbolic computation have not been verified. Formal methods can avoid these precision problems. Model checking and theorem proving are two commonly used formal methods. Considering the characteristics of fractional calculus, we use theorem proving to formally analyze the fractance element and fractance circuit. Theorem proving formalizes the systems and their properties into mathematical models, and then converts the mathematical models into logical models. It logically estimates the correctness of systems. Theorem proving is the strictest and most standardized method so far and the credibility of conclusion is also the highest. The theorem proof tool we use is HOL4. HOL system is one of the theorem prover and it is developed by Cambridge University. HOL4 is the fourth edition of HOL system and it is the newest edition. It is implemented basing on the meta-language. Meta-language is an interactive programming language and it is efficient and

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication.

This work was supported by the International S&T Cooperation Program of China (2010DFB10930, 2011DFG13000); the National Natural Science Foundation of China (60873006, 61070049, 61170304, 61104035, 61174145, 61201378); the Beijing Natural Science Foundation, Beijing Outstanding Talents Project, and the S&R Key Program of BMEC (4122017, KZ201210028036, KM201010028021, 2012D005016000011). Recommended by Associate Editor YangQuan Chen.

Shanshan Li, Chunna Zhao, Yong Guan, Zhiping Shi, Xiaojuan Li, Rui Wang, and Qianying Zhang are with the Capital Normal University. (email: shanshan_xiong@126.com, chunnazhao@163.com, guanyong@cnu.enu.cn, shizp@cnu.edu.cn, lixjxxxy@263.net, rwang04@gmail.com, zsjzqy@gmail.com)

Digital Object Identifier 10.1109/JAS.2016.7510208

strict. Now, HOL4 is widely used in the validation of software and hardware, and has obtained welcome results. Besides, HOL4 has a rich theorem library, including Boolean algebra, collection, Gauge integral, complex number and so on. The more theorems HOL4 has, the stronger the deduction ability is. Because a proof in the HOL4 system is constructed by repeatedly applying inference rules to axioms or to previously proved theorems.

As mentioned above, fractional calculus is the theoretical basis of fractance element, so we should formalize fractional calculus before the formal analysis of fractance element. On FMCAD2011, Umair and Osman [14] have formalized the Gamma function and the Riemann-Liouville definition of fractional calculus and formally verified some properties of them. And then they analyzed the fractional order behaviors of capacitance and differentiator. Their work pioneered the use of formal method for the analysis of fractional order systems. Shi Likun [15] has formalized the Grunwald-Letnikov definition of fractional calculus and formally analyzed the fractional order FC component and fractional position servo system in HOL4. In this paper, for the purpose of perfecting the definitions and properties of fractional calculus in HOL4, and improving the modelling and deduction ability of HOL4, we firstly establish the higher-order logic model of fractional calculus which is based on Caputo definition, and then formally verify some related basic properties of it. The formalization of theorem is known as the goal in HOL4 and we will use the existed definitions and theorems in HOL4 to prove the goal. It will illustrate the correctness of the theorem if the goal has been verified. We form these verified properties into separate theorems, so these definitions and theorems can be used directly by other users. At last, in order to illustrate the consistency of fractional calculus and integer order calculus and the validity of theorem proving method for the analysis of fractional order systems, we use the formalizations to formally analyze the fractance element and fractional differential circuit.

The rest of the paper is organized as follows: we present the formalizations of basic theories in Section 2, including the formalization of fractional calculus which is based on Caputo definition, and the verifications of some related basic properties. These basic properties include zero order property, the fractional differential of a constant and the consistency of fractional calculus and integer order calculus. In Section 3, the formalizations of these basic theories are applied to analyze fractance element and fractional differential circuit. The relationship of fractance element and ideal components, as well as the unification of fractional differential circuit and integer differential circuit are proved here. Section 4 concludes the paper.

II. Formalizations of Basic Theories

A. Caputo Definition of Fractional Calculus

The origin of fractional calculus can be traced back to more than 300 years. Fractional calculus is based on the definition of integer order calculus. It extends the order of integer order calculus from integer to non-integer. It can be used to describe actual systems more accurately. Grunwald-Letnikov, Riemann-Liouville and Caputo definition are the three commonly used definitions of fractional calculus. These three definitions have different characteristics. Grunwald-Letnikov definition is suitable for numerical computation while Riemann-Liouville definition which is defined in the form of differential-integral can make the mathematical analysis of fractional calculus become easier. Caputo definition can facilitate the modeling of actual problems and compact the Laplace transform of fractional calculus. The solution of fractional calculus equation is also given in the form of Caputo definition. In addition, Caputo definition is more able to reflect the feature that fractional calculus is the expansion of integer order calculus. Therefore, Caputo definition is more widely used in the modeling of actual problems [16].

These three definitions are defined from different perspectives. Riemann-Liouville definition and Caputo definition are the improvement of Grunwald-Letnikov definition. These three definitions can achieve unification under certain conditions. When the initial value is 0, the Grunwald-Letnikov definition and Caputo definition are equivalent. And the Riemann-Liouville definition and Caputo definition are equivalent when the original function $f(t)$ is $(\lfloor v \rfloor + 1)^{th}$ order derivable and all of the derivatives are 0, where v is the fractional order and the operator $\lfloor v \rfloor$ returns the biggest integer which is not greater than v.

In this paper, we research on the Caputo definition. The mathematical expression of fractional calculus based on Caputo definition is shown in Formula(1) [17].

$$ {}_a^C D_t^v f(t) = \frac{1}{\Gamma(m-v)} \int_a^t \frac{f^{(m)}(x)}{(t-x)^{v-m+1}} \, \mathrm{d}x. \quad m = \lfloor v \rfloor + 1 \tag{1} $$

where ${}_a^C D_t^v$ is the operator of fractional calculus with order v, lower limit a and upper limit t. Formula(1) is the unified expression of fractional differential and integral. When the order v is a positive value, Formula(1) means fractional differential and it means fractional integral when v is a negative value. The letter C on the top left corner of the operator is the abbreviation of Caputo. It indicates that Formula(1) is defined by Caputo definition, so that we can distinguish it from other definitions. Γ represents Gamma function[17] and its definition is as below.

$$ \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} \, \mathrm{d}t \tag{2} $$

where the real part of z is greater than 0. Gamma function is the most commonly used basic function of fractional calculus. It extends the factorial from a natural number to a real number. Gamma function is also known as generalized factorial.

When modeling and verifying fractance element in HOL4, the formal model of fractional calculus based on Caputo definition is needed. We firstly establish the formal model of fractional calculus based on Caputo definition in HOL4.

Definition 1. Fractional Calculus based on Caputo Definition

$\forall f\ v\ a\ t.frac\_c\ f\ v\ a\ t\ =\ if\ (v=0)\ then\ f\ t$
$\quad else$
$\lim(\lambda n.1/Gamma\ (\&(flr\ v)+1-v)*(integral(a,t-1/2\ pow\ n)(\lambda x.(((t-x)\ rpow\ (\&(flr\ v)+1-v-1))*(n\_order\_deriv(flr\ v+1)f\ x)))))$

Definition 1 is formalized basing on the real library[18], transcendental function library and integer order integral library[19] which have been already formalized in HOL4. The operator $frac\_c$ represents the Caputo definition of fractional calculus. f is the initial function of type $(real- > real)$. v is a real number which indicates the order of fractional calculus. t and a are the upper and lower limit, respectively. $Gamma$ represents Gamma function which has been formalized in reference[14]. $(flr\ v)$ is the formalization of $\lfloor v \rfloor$. $integral(a, t-1/2\ pow\ n)$ represents integral with lower limit a and upper limit $(t - 1/2\ pow\ n)$ where pow is a power function with natural exponent.

Formula(1) is a definite integral with variable upper limit. The formalization of variable upper limit is a difficulty for our work. The function $integral(a, b)f$ in HOL4 only represents integral with lower limit a and upper limit b, where a and b are both constant. The variable upper limit should be reconstructed according to the existing definitions and theorems in HOL4. We solve this problem by nesting the integral into the limitation. Taking the existing definitions and theorems into account, we construct formula $(t - \frac{1}{2^n})$ and take the limit of it as $\lim_{n\to+\infty}(t - \frac{1}{2^n})$. The variable upper limit t will be expressed when $\frac{1}{2^n}$ becomes very close to 0 as n becomes very large. Here we use $lim(\lambda n.t - 1/2\ pow\ n)$ to formalize the variable upper limit t in HOL4. $lim(\lambda n.f)$ computes the limit of f when n tends to infinity and it is a function in sequence library [20].

Caputo definition has certain requirements for the original function. As can be seen from Formula(1), it firstly requires the original function $f(x)$ is m^{th} order derivable. Moreover, the product of $f^{(m)}(x)$ and $\frac{1}{(t-v)^{v-m+1}}$ should be integrable. Besides, in practical applications of fractional calculus such as fractance element, their parameters are always based on time so that we can analyze the systems from one moment to the next moment. So, the upper limit and lower limit which is based on time here should satisfy the condition that the upper limit should be greater than lower limit. Furthermore, we stipulate Formula(1) is limited because a limitation is used to denote the variable upper limit in the formalization of Caputo definition. These existent conditions of Caputo definition are formalized as follows:

Definition 2. Existent Conditions
$\forall f\ v\ a\ t\ n\ l.frac\_c\_exists\ f\ v\ a\ t\ n\ l\ =\ (\forall m.m <= (flr\ v{+}1) ==> (\lambda t.n\_order\_deriv\ m\ f\ t)\ differentiable\ t)$
$\wedge\ (\forall v.integrable(a, t-1/2\ pow\ n)(\lambda x.(((t-x)\ rpow$
$(\&(flr\ v){+}1{-}v{-}1)){*}(n\_order\_deriv(flr\ v{+}1)\ f\ x)))) \wedge$
$a <= t-1/2\ pow\ n \wedge (\lambda n.1/Gamma\ (\&(flr\ v)+1-v)*$
$(integral(a, t-1/2\ pow\ n)(\lambda x.(((t-x)\ rpow\ (\&(flr\ v)+$
$1-v-1))*(n\_order\_deriv(flr\ v+1)\ f\ x))))) -- > l)$

Only the above conditions are met, the Caputo definition and its formalization are existent. When using operator ${}^C_aD^v_t$, we always assume that these existent conditions are established. The formalization of these conditions can be utilized to be the antecedent when proving the subsequent properties of fractional calculus.

B. Zero Order Property

If function $f(t)$ satisfies the existent conditions of Caputo definition and the order of fractional calculus is 0, the fractional calculus of $f(t)$ will return the original function. The property is shown in Formula(3).

$$ {}^C_aD^0_tf(t) = f(t) \tag{3} $$

The formal verification of this property in HOL4 is given in the following theorem.

Theorem 1. Zero Order Property
$\forall f\ v\ a\ t\ n.$
$frac\_c\_exists\ f\ v\ a\ t\ n\ l ==> (frac\_c\ f\ 0\ a\ t = f\ t)$
where $frac\_c$ and $frac\_c\_exists$ have been formalized in Definition 1 and Definition
2. A special case that the order is 0 has been considered in Definition 1, so the proof of Theorem 1 is relatively simple. There are two proving methods in HOL4 system, including forward proof and goal oriented proof. The second method is more commonly used. In this paper, we use the method of goal oriented proof. This method uses the tactic of HOL4, and the existing conditions, definitions and theorems to divide the original goal into one or more relatively simple sub-goals. Then we only have to prove these sub-goals. And the original goal will be proved when all the sub-goals are proved. In the proof of Theorem 1, the combination of tactic $REPEAT$ and tactic GEN_TAC is used to remove all of the universal quantifiers firstly. And then the proof is completed by using Definition 1 to rewrite the current goal.

C. Fractional Differential of a Constant

$$ {}^C_aD^v_tC = \begin{cases} C, & v = 0 \\ 0, & v > 0 \end{cases} \tag{4} $$

Caputo definition is commonly used in engineering applications. This is partly because the fractional differential of a constant under Caputo definition is bounded, as shown in Formula(4), while it is unbounded under other definitions. For example, under Riemann-Liouville definition, the fractional differential of a constant C is expressed as ${}^{RL}_aD^v_tC = \frac{Ct^{-v}}{\Gamma(1-v)}$, which will be bounded unless the starting point t tends to ∞. However, it is impossible to set the starting point to $-\infty$ when analyzing the transient process. Hence, Caputo definition is more appropriate in engineering applications[21]. The formal verification of this special property in HOL4 is given in Theorem 2.

Theorem 2. Fractional Differential of a Constant
$\forall f : real- > real\ c : real\ v : real\ a\ t.$
$(\forall a\ t\ n\ l.frac\_c\_exists\ f\ v\ a\ t\ n\ l) \wedge (0 <= v) ==>$
$(frac\_c\ (\lambda t.c)\ v\ a\ t = if\ (v = 0)\ then\ c\ else\ 0)$

The integer order derivative of a constant is included in the verification of Theorem 2. In order to simplify the verification process and facilitate the formal verification of other verification, here we firstly verify the integer order derivative of a constant and form it as a separate lemma.

Lemma 1. Integer Order Derivative of Constant c
$\forall m\ c.(0 < m) ==> (n\_order\_deriv\ m\ (\lambda x.c)\ t = 0)$

Lemma 1 verifies that the m^{th} order derivative of constant c is 0. This result is consistent with the mathematical result. The variable m in Lemma 1 is a positive integer and it has infinite possibilities. For such goal, we generally use mathematical induction method to prove it. The proof of Lemma 1 is finished by using mathematical induction method twice. We firstly make an induction on m and then divide the goal into two cases whose precondition is $(0 < 0)$ and $(0 < m+1)$, respectively. As we all know, the premise condition $(0 < 0)$ is not established. The inference rule of HOL4 is that any conclusion can be deduced by the impossible precondition. Here we use the tactic $FULL\_SIMP\_TAC$ to complete the proof of the first case. In the proof of the second case, we firstly verify that the $(m+1)^{th}$ order derivative of c is equal to the m^{th} order derivative of the derivative of c, and then prove that the derivative of c is equal to 0. Finally, the proof of the second case can be done by doing a mathematical induction on m again. Hence Lemma 1 is proved.

Theorem 2 is implemented with statement $if \cdots then \cdots else \cdots$ because it has two cases. One case is that the differential order is 0 and the other one is that the differential order is greater than 0. We firstly proceed with Theorem 2 by separating the goal into two sub-goals using tactic $COND\_CASES\_TAC$.

The first sub-goal describes that the fractional differential of a constant returns the constant itself when the differential order is 0. We finish the proof of the first sub-goal by using an assumption and Theorem 1 to rewrite the goal.

For the case that the differential order is greater than 0, we firstly use tactic $COND\_CASES\_TAC$ to divide the present goal into two sub-goals:

$C = 0$

and

$lim(\lambda n.1/Gamma\ (\&flr\ v + 1 - v) * integral(a, t - 1/2\ pow\ n)(\lambda x.(t - x)\ rpow\ (flr\ v + 1 - v - 1) * n\_order\_deriv\ (flr\ v + 1)\ (\lambda t.C)\ x)) = 0.$

For the sub-goal $C = 0$, C is an arbitrary constant so we cannot say that C must equal to 0. But there is a contradiction between $(v \neq 0)$ and $(v = 0)$ in the assumption. According to the inference rule of HOL4, we apply tactic $FULL\_SIMP\_TAC$ to deduce sub-goal $C = 0$. In the proof of the second sub-goal, we firstly establish a new sub-goal:

$\forall n\ c\ x.n\_order\_deriv\ (flr\ v + 1)\ (\lambda t.c)\ x = 0$

It can be seen that the new sub-goal is the conclusion in Lemma 1. We can directly apply Lemma 1 to prove the above sub-goal as long as we can prove that the order $(flr\ v + 1)$ is greater than 0. It is difficult to prove $(0 < flr\ v + 1)$. Here, the assumption is used to deduce that v is greater than 0 firstly. Secondly, we prove that $(flr\ v)$ is equal to or greater than zero by using theorems NUM_FLOOR_LE2 and $REAL\_LT\_IMP\_LE$. Thirdly, it can be naturally proved that $(flrv + 1)$ is greater than 0 by using the combination of theorem $GSYM\ ADD1$ and tactic $REWRITE\_TAC$ as well as theorem $LESS\_EQ\_IMP\_LESS\_SUC$ and tactic $FULL\_SIMP\_TAC$. Now that Lemma 1 can be used to deduce the above sub-goal. Then the proved sub-goal can be applied to simplify the original goal. Now, the second sub-goal of Theorem 2 is simplified to:

$lim(\lambda n.1/Gamma\ (\&flr\ v + 1 - v) * integral(a, t - 1/2\ pow\ n)(\lambda x.(t - x)\ rpow\ (\&flr\ v + 1 - v - 1) * 0)) = 0$

Then the item $integral(a, t - 1/2\ pow\ n)(\lambda x.(t - x)\ rpow\ (\&flr\ v + 1 - v - 1) * 0))$ is simplified to $(integral(a, t - 1/2\ pow\ n)(\lambda x.0))$ by using theorem $REAL\_MUL\_RZERO$. Next, we prove that $(lim(\lambda n.1/Gamma\ (\&flr\ v + 1 - v) * 0))$ is equal to $(lim(\lambda n.0))$. With this, the second sub-goal of Theorem 2 is simplified to:

$$lim(\lambda n.0) = 0$$

Finally, the formal verification of Theorem 2 is done by using the definition lim, theorem $INTEGRAL\_CONST$ and theorem SEQ_CONST.

D. Integer Order Differential is the Special Case of Fractional Differential

Fractional differential is the generalized form of integer order differential and integer order differential is the special case of fractional differential. When the order m is a positive integer and the initial condition is 0, fractional differential is consistent with integer order differential. Theorem 3 is the formal verification of this property. Lemma 2 and Lemma 3 are the required lemmas in the verification of Theorem 3. We also prove these two lemmas separately.

Theorem 3. Integer Order Differential is the Special Case of Fractional Differential
$\forall f\ m\ n\ t.(0 <= m \wedge (\forall t\ n.frac\_c\_exists\ f\ (\&m)\ a\ t\ n\ l) \wedge ((n\_order\_deriv\ m\ f\ a) = 0)) ==> (frac\_c\ f\ (\&m)\ a\ t = n\_order\_deriv\ m\ f\ t)$

Lemma 2. The Derivative of nth Order Derivative is (n+1)th Order Derivative
$\forall m\ f\ t.(\forall m.m <= n + 1 ==> (\lambda t.n\_order\_deriv\ m\ f\ t)\ differentiable\ t) ==> ((\lambda t.n\_order\_deriv\ n\ f\ t)\ diffl\ (n\_order\_deriv\ (n + 1)\ f\ t))\ (t)$

Lemma 3. Newton Leibniz Formula
$\forall (f : real- > real)\ (f' : real- > real)\ a : real\ b : real.\ a <= b \wedge (\forall t.a <= t \wedge t <= b ==> (f\ diffl\ f'\ t)\ t) ==> (integral(a, b)\ f' = f\ b - f\ a)$

Lemma 2 shows that equation $\frac{df^n(t)}{dt} = f^{n+1}(t)$ will be tenable if function $f(t)$ is $(n+1)^{th}$ order derivable. Lemma 3 verifies that the integral of function f' in interval [a,b] equals to the difference value between the value of function f at upper limit and the value of function f at lower limit, where f is the original function of f'. Proofs of these two lemmas are challenging for us. The key is to transform the goal flexibly. When proving Lemma 2, we are unable to do a further conversion of the goal until we change our way of thinking. The precondition of Lemma 2 is that function f is m^{th} order derivable for every m which meets condition $(m \leq n+2)$. Taking the definition of m^{th} order derivative into account, we convert the precondition to that f' is m^{th} order derivative for every m which meets condition $(m \leq n + 1)$, where f' is the derived function of f. This conversion enables the proof to be applied with the definition of m^{th} order derivative, and then we can overcome the difficulty. Analogously, in the proof of Lemma 3, we need to prove a sub-goal $(n\_order\_deriv\ 1\ f\ x = deriv\ f\ x)$

which is also unable to do further transformation under the existing theorems. Here, if the number 1 is replaced with $(SUC\ 0)$ which means $(0 + 1)$, it will be possible for us to use the definition of m^{th} order derivative to rewrite the sub-goal and then finish the proof.

In Theorem 3, the antecedent (0≤m) limits that Formula(1) just indicates fractional differential. $(n\_order\_deriv\ m\ f\ a = 0)$ denotes that the initial condition of fractional calculus is 0. HOL4 is a rigorous tool for logical verification and error will occur if the type is not consistent. Here, the order m is a natural number of type num while the order defined in $frac\_c$ needs to be the type of $real$. So we should introduce operator & to map the natural number m to its corresponding real number of type $real$ here. Only in this way, we can avoid the error of type inconsistency.

The formal verification of Theorem 3 is relatively complex. We firstly simplify Theorem 3 to two sub-goals by using Definition 1 and tactic RW_TAC:

$f\ t = n\_order\_deriv\ m\ f\ t$
$----------0.\exists t\ n.frac\_c\_exists\ f\ (\&m)\ a\ t\ n\ l$
$1.n\_order\_deriv\ m\ f\ a = 0$
$2.\&m = 0$

and

$lim(\lambda n.$
$1/Gamma\ (\&flr(\&m) + 1 - \&m) *$
$integral(a, t - 1/2\ pow\ n)$
$(\lambda x.$
$(t - x)\ rpow\ (\&flr\ (\&m) + 1 - \&m - 1) *$
$n\_order\_deriv\ (flr\ (\&m) + 1)\ f\ x)) =$
$n\_order\_deriv\ m\ f\ t$
$----------0.\exists t\ n.frac\_c\_exists\ f\ (\&m)\ a\ t\ n\ l$
$1.n\_order\_deriv\ m\ f\ a = 0$
$2.\&m <> 0$

What should be mentioned here is that the statements under imaginary line are the assumptions which are the known conditions of the goal. For the first sub-goal, we firstly prove $(m = 0)$ by using theorem $GSYM\ REAL\_INJ$ and the known conditions. Then the proof of the first sub-goal can be done by utilizing the definition $n\_order\_deriv\_def$. In the proof of the second sub-goal, we firstly deduce another condition (0≤&m) from the known condition (&m<>0) by applying the statement $(ASSUM\_LIST(fn\ thl\ => ASSUME\_TAC(REWRITE\_RULE[REAL\_LT\_NZ]$ $(el\ 3\ (rev\ thl)))))$ to the current goal. Here, $ASSUM\_LIST(fn\ thl)$ represents the operation on the assumption list, $ASSUME\_TAC$ and $REWRITE\_RULE$ are the tactics of HOL4, $REAL\_LT\_NZ$ is a theorem and $(el\ 3\ (rev\ thl))$ is a positioning statement. Next, we establish a new sub-goal:

$\forall n\ x.(\&(flr((\&(m : num)) : real) : num) : real) = \&m$
And the above new sub-goal can be verified by using theorem $REAL\_IN$ and NUM_FLOOR_EQNS. Then we utilize the proved sub-goal to simplify the initial goal to:

$lim(\lambda n.1/Gamma\ (\&m + 1 - \&m) * integral(a, t - 1/2\ pow\ n)(\lambda x.(t - x)\ rpow\ (\&m + 1 - \&m - 1) * n\_order\_deriv\ (flr\ (\&m) + 1)\ f\ x)) = n\_order\_deriv\ m\ f\ t$

The next step is to simplify $(flr\ (\&m))$ to m by using theorem $REAL\_INJ$. Then we use theorem $REAL\_ADD\_SUB$ and $REAL\_SUB\_REFL$ to prove that $(\&m + 1 - \&m - 1)$ is equal to 0. Next, the sub-goal $(\forall n\ x.(t - x)\ rpow\ 0 * n\_order\_deriv\ (m + 1)\ f\ x = n\_order\_deriv\ (m + 1)\ f\ x)$ is verified and used to simplify the second sub-goal of Theorem 3 to: $lim(\lambda n.1/Gamma(1) * integral(a, t - 1/2\ pow\ n)(\lambda x.n\_order\_deriv\ (m + 1)\ f\ x)) = n\_order\_deriv\ m\ f\ t$

The property of Gamma function $GAMMA\_1\_EQUAL\_1$ which has been verified in reference [14] is used here to verify that $(Gamma\ 1)$ is equal to 1. Finally, we accomplish the proof of Theorem 3 by using Lemma 2, Lemma 3, the known conditions and the definition and properties of limit function.

Establishing sub-goal is needed in the proving process many times. But when we use the proved sub-goal to rewrite the goal, it fails. For instance, when proving goal $(lim(\lambda n.1/Gamma(1) * integral\ (a, t - 1/2\ pow\ n)\ (\lambda x.(t - x)\ rpow\ 0 * n\_order\_deriv\ (m + 1)\ f\ x)) = n\_order\_deriv\ m\ f\ t)$, we establish a sub-goal $((t - x)\ rpow\ 0 * n\_order\_deriv\ (m + 1)\ f\ x = n\_order\_deriv\ (m + 1)\ f\ x)$ and then prove it. The types of variables in the sub-goal are completely consistent with the types in initial goal, but it fails when we use the sub-goal to simplify the initial goal. This is because (λn) in the initial goal has the implication of arbitrary n. We overcome this problem by adding (λn) to the sub-goal when we established it.

E. Integer Order Integral is the Special Case of Fractional Integral

Similarly, when the order of fractional calculus is a negative integer m, the fractional integral ${}_a^C D_t^m$ is the same as the mth order integral of integer order calculus. When the order of fractional calculus is -1, there is Formula(5).

$$ {}_a^C D_t^{-1} f(x) = \int_a^t f(x)\mathrm{d}x \tag{5} $$

We formally verify Formula(5) in HOL4 as follows:

Theorem 4. First Order Integral is the Special Case of Fractional Integral
$\forall f\ a\ t.FLR\_NEG\_1 \wedge FLR\_NEG\_0 ==> (frac\_c\ f$ $(-\&(1 : num))\ a\ t = lim(\lambda n.integral(a, t-1/2\ pow\ n)\ f))$

Definition of FLR_NEG_1 and FLR_NEG_0 are respectively shown as follows:
$FLR\_NEG\_1 = (\&flr\ (-\&(1 : num)) = -1)$
$FLR\_NEG\_0 = (flr\ (-\&(1 : num)) + 1 : num = 0 : num)$

The first definition indicates that -1 is round off to -1. The second definition demonstrates that the result of the rounding off of -1 plus 1 is 0. When proving Theorem 4, we firstly use Definition 1 to rewrite the goal and then utilize tactic $COND_CASES_TA$C to divide the goal into two sub-goals:

$ft = lim(\lambda n.integral\ (a, t - 1/2\ pow\ n)\ f)$

and

$lim(\lambda n.1/Gamma\ (\&flr\ (-1) + 1 - -1) * integral(a, t - 1/2\ pow\ n)(\lambda x.(t - x)\ rpow\ (\&flr\ (-1) + 1 - -1 - 1) * n\_order\_deriv\ (flr\ (-1) + 1)\ f\ x)) = lim(\lambda n.integral\ (a, t - 1/2\ pow\ n)\ f)$

According to the inference rules of HOL4, the first sub-goal can be deduced by the contradictory assumptions. For the second sub-goal, we firstly use theorem $REAL\_SUB\_RNEG$ to simplify $(1 - -1)$ to $(1 + 1)$. Then we utilize the above two definitions and some laws of computing to simplify the current goal. Finally, the definition of m^{th} order derivative and theorem ETA_THM are used to realize the proof.

Proofs of properties not only ensure the correctness of the formalization of fractional calculus based on Caputo definition, but also reduce the interventions of user when formally analyze the fractional order systems. The formalizations of fractional calculus and its properties are the keys to formally analyze fractional order systems. The work in this section provides the bases to the formal analysis of fractance element and fractional differential circuit.

III. Formalization of Fractance Element

The actual circuits tend to show the fractional order behavior. For example, the modeling and analysis of traditional capacitance is always based on the integer order differential theory. However, with the development of nonlinear theory and fractal differential geometry theory, the researchers found that the traditional capacitance which is based on integer order calculus is just the idealization of the actual model. Actually, the ideal capacitance does not exist in practice. This is mainly because that the electrolyte materials which make up the capacitance show the fractal dimension characteristic. In fact, the capacitance presents the fractional characteristics on the physical property. The integer order differential and integral circuit which respectively show the behaviors of high and low pass, are also the results of the idealized processing of real circuit. Fractance element and fractional differential circuit can describe the fractional order behaviors of circuits. We will use fractional calculus to model them. Then the relationship of fractance element and ideal elements as well as the unification of fractional differential circuit and integer differential circuit are verified. The purpose is to illustrate that fractional calculus is the extension of integer order calculus and the real systems can be described more comprehensively by using fractional calculus. Meanwhile, the correctness of the above formalizations and the effectiveness of theorem proving method in the analysis of fractional order systems are also demonstrated here.

Any lumped parameter element can be described by mathematical model and physical model. Fractance element is no exception. In this section, we will give the circuit symbol graph and mathematical expression of fractance element, and then formally analyze it and fractional differential circuit which is the simplest circuit composed by fractance element.

Fractance element is a two-port element and it can be represented by symbol F. The circuit symbol graph of it is given in Fig.1.

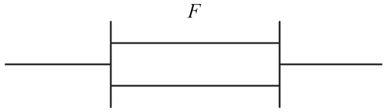


Fig. 1. Fractance Element

In the complex frequency domain, the impedance of fractance element is shown as $Z(S) = kS^v$, where k is a constant coefficient and v is the order of fractance element. In reference [1], when the order v is greater than 0, fractance element which has the form of $Z(S) = kS^{-v}$ is called fractional capacitance and it is called fractional inductance when it is in the form of $Z(S) = kS^v$. The voltage and current of fractance element satisfy Ohm's law and the relationship of them is given in Formula (6).

$$i(t) = kD^v[v(t)] \tag{6}$$

As can be seen from Formula (6), the current of fractance element is the v^{th} order calculus of voltage and v is a real number. Due to the arbitrariness of order v, the definition of fractance element is broader than common components and the function of fractance element is also more powerful. Definition 3 is the formalization of fractance element in HOL4.

Definition 3. Fractance Element
$\forall k\ v\_t\ v\ t.i\_t\ k\ v\_t\ v\ t = k * frac\_c\ v\_t\ v\ 0\ t$
where i_t and v_t are the current and voltage of fractance element at moment t, respectively. k is the constant coefficient and $frac\_c$ is the formalization of fractional calculus based on Caputo definition which is given in Definition 1.

According to the circuit analysis theory, the current flowing through the resistance R is $i(t) = \frac{v(t)}{R}$, while the current flowing through the capacitance C is $i(t) = C\frac{dv(t)}{dt}$ and the current flowing through the inductance L is $i(t) = \frac{\int_0^t v(\tau)d\tau}{L}$. Introducing the concept of fractance element, the resistance can be understood as the case where the order of fractance element is 0, and the capacitance can be understood as the case where the order is 1 and the inductance can be understood as the case where the order is -1. Therefore, there are connections between the fractance element and the traditional resistance, inductance and capacitance. The traditional components are three ideal models of actual components. Fractance element can better describe the performance of practical elements in the circuit. Based on Definition 3, we will use the formalizations in Section 2 to formally verify the relationship between fractance element and the three ideal components. The formal verifications of these three relationships using higher-order logic are shown below.

Theorem 5. Relationship between Fractance Element and Resistance
$\forall k\ v\_t\ t.(v = 0 : real) ==> (i\_t\ k\ v\_t\ v\ t = k * v\_t\ t)$

Theorem 6. Relationship between Fractance Element and Capacitance
$\forall k\ v\_t\ t.(v = \&(1 : num)) \wedge (\forall v\ t\ n\ l\ a.$
$frac\_c\_exists\ v\_t\ v\ a\ t\ n\ l) \wedge (n\_order\_deriv\ 1\ v\_t\ 0 =$
$0) ==> (i\_t\ k\ v\_t\ v\ t = k * deriv\ v\_t\ t)$

Theorem 7. Relationship between Fractance Element and Inductance
$\forall k\ v\_t\ t.(v = -\&(1 : num)) \wedge FLR\_NEG\_1 \wedge$
$FLR\_NEG\_0 ==> (i\_t\ k\ v\_t\ v\ t = k *$
$lim(\lambda n.integral(0, t - 1/2\ pow\ n)\ v\_t))$

Theorem 5 verifies that the fractance element exhibits the behavior of a resistance for $v = 0$. Here, the relationship between the flowing current and voltage is expressed as

$i(t) = kv(t)$ in time domain, where k equals to $\frac{1}{R}$ and R is the value of resistance. The proof of Theorem 5 is completed by using Definition 3 and Theorem 1 to rewrite the goal and make a further calculation. Theorem 6 proves that fractance element displays the characteristic of ideal capacitance for $v = 1$. At this time, the relationship between the flowing current and voltage is expressed as $i(t) = k\frac{dv(t)}{dt}$, where k means the value of capacitance. The formal verification of Theorem 6 is based on Theorem 3. Theorem 7 deduces that fractance element will behave as an ideal inductance if $v = -1$. The connection between the flowing current and voltage is expressed as $i(t) = k\int_0^t v(\tau)d\tau$, where k equals to $\frac{1}{L}$ and L is the value of inductance. Theorem 4 is utilized in this proof.

The fractional differential circuit is a kind of fractance circuit and it is composed of fractance element. It outputs the fractional differential of input signal and its amplitude frequency characteristic is a high pass filter. In terms of fractional order controller, initial implementation of fractional differential circuit[22] makes a foundation for the universal application of fractional order controllers in the field of information science[23]. Fig.2 is a fractional differential circuit with power source Vi, resistance R and fractance element F which is realized by fractional capacitance. Based on the formalization of fractance element, the formal modeling and verification of fractional differential circuit will be performed next.

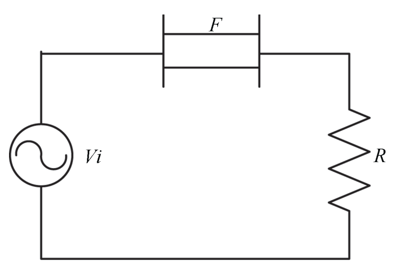


Fig. 2. Fractional Differential Circuit

The output voltage of fractional differential circuit in Fig.2 is the voltage across the resistance R and the input voltage is the voltage of power source. The relationship between the output voltage and the input voltage is inferred as:

$$v_o(t) = RCD^v v_i(t) \qquad (v > 0) \tag{7}$$

where $v_o(t)$ is the output voltage and $D^v v_i(t)$ returns the v^{th} order calculus of input voltage $v_i(t)$. The condition of Formula(7) limits the operator $D^v v_i(t)$ as the expression of fractional differential. The order v is the same as the order of fractance element. The formalization of fractional differential circuit in HOL4 is given in Definition 4.

Definition 4. Fractional Differential Circuit
$\forall R\ C\ vi\_t\ v\ t.vo\_D\_t\ R\ C\ vi\_t\ v\ t = R * C * frac\_c\ vi\_t\ v\ 0\ t\ R\ C\ vi\_t\ v\ t.vo\_D\_t\ R\ C\ vi\_t\ v\ t = R * C * frac\_c\ vi\_t\ v\ 0\ t$
where vo_D_t and vi_t indicate the output voltage and the input voltage of the circuit at moment t. vo_D_t and vi_t are both type of $(real- > real)$ here. R, C, v and t represent resistance, capacitance, differential order and the upper limit. If the order v equals to 1, Definition 4 will represent a first order differential circuit. For the first order differential circuit, the output response just reflects the rate of input change. So the output response of first order differential circuit is 0 if a constant signal is applied at the input. This is because the rate of change for constant signal is 0. The following is the verification of this property in HOL4 using the already verified definition and properties in Section 2.

$(v = 1) \wedge (\exists a\ t\ n\ l.frac\_c\_exists\ (\lambda t.v\_0 : real)\ v\ a\ t\ n\ l)$
$==> (vo\_D\_t\ R\ C\ (\lambda t.v\_0 : real)\ v\ t = 0)$

The precondition $(v = 1)$ guarantees that the order of fractional differential circuit is 1. Under this condition, fractional differential circuit will behave as first order differential circuit of integer order calculus. The second precondition $(frac\_c\_exists\ (\lambda x.v\_0 : real)\ v\ a\ x\ n\ l)$ ensures the existence of fractional calculus which is based on Caputo definition for function v_0 . Under these two preconditions, it can be gradually verified that the output response of this fractional differential circuit is 0 when the constant signal v_0 is the input. The availability of already verified property of fractional calculus in Section 2 let us to achieve the simple sub-goal.

The fractional order differentiator has been formalized in reference[12]. It formally verified the output response of fractional order differentiator when unit step signal is applied at the input and the order is between 0 and 1. A lot of work has been done in [12], which is very significant and gives us much inspiration. However, the formal verification of fractional order differentiator did not take the order of integer 1 into account. In other words, they have not considered the unification of fractional differential and integer order differential. In this paper, we take the differential order of integer 1 into account, and use the Caputo definition to verify the output response of the fractional differential circuit with constant signal v_0 as input. The fractional differential circuit will behave as first order differential circuit if the order is integer 1. According to the property of Riemann-Liouville definition of fractional calculus, the output response of first order differential circuit is RCv_0t^{-1} when constant signal v_0 is applied at the input. This result is in contradiction with the result that the output response of first order differential circuit only reflects the rate of input change. As analyzed above, the result using Caputo definition in this paper is consistent with the fact. The formal result not only verifies the consistency of fractional first order differential circuit and integer first order differential circuit, which besides achieving uniformity in fractional differential and integer order differential, but also correctly deduces the output response of first order differential circuit with constant signal as the input.

Due to the completeness of theorem proving, the results are accurate and complete. Besides, the results in this paper are consistent with the theoretical results, which illustrate the correctness of the formalizations of fractional calculus, as well as the validity of the analysis of fractance element using theorem proving.

IV. CONCLUSION

Theorem proving, as a formal method, formalizes the specifications and designs of systems to the logic models. The validation process is intuitional and rigorous. Besides, its self-prove function can ensure the correctness of formalization. Based on theorem proof tool HOL4, we completed the formalized analysis of fractance element and fractional differential circuit which is made up of fractance element. Caputo definition of fractional calculus and some properties of it are the theoretical bases of the formal analysis of fractance element and fractional differential circuit. Therefore, their formalization is a significant work presented in this paper. These works factually lay good foundations for the formal analysis of circuits fractional order behaviors. Meanwhile, the formal analysis of fractance elements and fractional differential circuit also shows the effectiveness, practicality and correctness of the formalizations of fractional calculus theorems. The formalization of fractional calculus based on Caputo definition in this paper, completes the definition of fractional calculus in HOL4 and provides more choices for the formal analysis of fractional order systems. In addition, the formal analysis of fractance element not only enriches the studies of fractance element, but also provides a way to the analysis of fractance element. The next step will be taken to verify the other properties of fractional calculus based on Caputo definition, to lay a solid foundation for the complete analysis of fractance element and fractional differential circuit.

REFERENCES

[1] hou Ji-Liu, Pu Yi-Fei, Liao Ke. *Fractional Calculus's Principle and its Application in Signal Analysis and Processing*. Beijing:Science Press, 2006. 151–154

[2] Zhang YZ. Study on Fractional-order Calculus and its Applications [Ph.D.dissertation], Northeastern University, China, 2008

[3] Deng Li-Wei, Song Shen-Min. Synchronization of fractional order hyperchaotic systems based on output feedback sliding mode control. *Acta Automatica Sinica*, 2014, **40**(11):2420–2427

[4] Liao Guangkai, Long Zhilin, Xu Fu, et al. Investigation on the viscoelastic behavior of an fe-base bulk amorphous alloys based on the fractional order rheological model. *Acta Physica Sinica*, 2015, **64**(13): 136101–136101

[5] Mehdi Dalir. Applications of fractional calculus. *Applied Mathematical Sciences*, 2010, **4**(21): 1021–1032

[6] Huang Guo, Xu Li, Pu Yi-Fei. Summary of research on image processing using fractional calculus. *Application Research of Computers*, 2012, **29**(2):414–426

[7] Zafer BINGUL, Oguzhan KARAHAN. Fractional PID controllers tuned by evolutionary algorithms for robot trajectory control. *Turkish Journal of Electrical Engineering & Computer Sciences*, 2012, **20**(Sup.1), 1123–1136

[8] Yan Li-Mei, Zhu Yu-Song, Xu Jian-Jun, et al. Transmission lines modeling method based on fractional order calculus theory. *Transactions of China Electrotechnical Society*, 2014, **29**(9):260–268

[9] Cao Jun-Yi, Liang Jin, Cao Bing-Gang. Fuzzy fractional order controller based on fractional calculus. *Journal of Xi'an Jiao Tong University*, 2005, **39**(11):1246–1253

[10] Krishnan Balachandran, Venkatesan Govindaraj, Luis Rodrłguez-Germ, et al. Stabilizability of fractional dynamical systems. *Fractional Calculus & Applied Analysis*, 2014, **17**(2):511–531

[11] Zhang Song. Implementation of electric circuit for changeable order-changeable fractional resistance. *Communication Technology*, 2010,**43**(10):164–166

[12] Zhou Ji-Liu, Yuan Xiao, Liao Ke, et al. One Method to realize alterable order analog fractance circuits . *Journal of Sichuan University (Engineering Science Edition)*, 2007, **39**(3):141–144

[13] Ren Yi, Yuan Xiao. Implementation of fractional-order analog fractance circuit using binomial expansion. *Journal of Sichuan University (Natural Science Edition)*, 2008,**45**(5):1100–1104

[14] Umair Siddique, Osman Hasan. Formal analysis of fractional order systems in HOL. *Formal Methods in Computer-Aided Design*, 2011.163–170

[15] Shi LK. The Formal Analysis of Fractional Systems based on Grunwald-Letnikov Definition using Higher-order Logic[Master dissertation], Capital Normal University, China, 2014

[16] Zivorad Tomovski, Roberto Garra. Analytic solutions of fractional integro-differential equations of volterra type with variable coefficients. *Fractional Calculus & Applied Analysis*, 2014, **17**(1):38–60

[17] Zhao Chun-Na, Li Ying-Shun, Lu Tao. *The Analysis and Design of Fractional Systems*. Beijing: National Defence Industry Press, 2010.13–20

[18] Harrison J. *Theorem Proving with the Real Numbers*. Berlin:Springer-Verlag, 1998

[19] Gu Wei-Qing, Shi Zhi-Ping, Guan Yong, et al. Formalization of gauge integration theory in HOL4, *Computer Science*,2013, **40**(2):191–194

[20] Zhao Gang, Zhao Chun-Na, Guan Yong, et al. Formalization of laplace transform calculus in HOL4. *Micro Computer System*, 2014, **35**(9):2178–2181

[21] Mohammad Saleh Tavazoei. Time response analysis of fractional-order control systems:a survey on recent results. *Fractional Calculus & Applied Analysis*, 2014, **17**(2):440–461

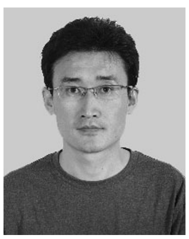
[22] M.Nakagava, K.Sorimachi. Basic characteristics of a fractance device. *IEICE TRANSACTIONS on Fundamentals of Electronics, Communications and Computer Sciences*, 1992, **E75-A**(12):1814–1819

[23] Wang Ji-Feng. *The Analysis of Control Performance for Fractional-order Systems*. Beijing: Electronic Industry Press, 2010.13–14

LI Shanshan is currently a graduate student in Capital Normal University. She received her bachelor degree from Minzu University of China in 2013. Her research interests cover formal verification and fractional calculus.
e-mail:shanshan_xiong@126.com

ZHAO Chunna graduated from Northeastern University in 2006 and received the Ph.D degree. She is currently an associate professor in Capital Normal University. Her research interests include formal verification and the modeling and control of fractional order system.
e-mail:chunnazhao@163.com
corresponding author

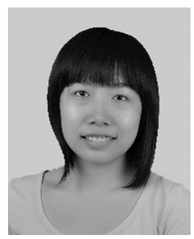
GUAN Yong graduated from China University of Mining and Technology in 2004 and received the Ph.D degree. He is currently a professor in Capital Normal University. His research interests cover formal verification, robot and the embedded system with high reliability.
e-mail:guanyong@cnu.edu.cn



SHI Zhiping graduated from Institute of Computing Technology, Chinese Academy of Sciences in 2005 and received the Ph.D degree. He is currently an associate researcher in Capital Normal University. His research interests include formal verification and artificial intelligence.
e-mail:shizp@cnu.edu.cn

LI Xiaojuan graduated from China Agricultural University in 1999 and received the Ph.D degree. She is currently a professor in Capital Normal University. Her research interests cover formal verification and the computer network.
e-mail:lixj@cnu.edu.cn

WANG Rui graduated from Tsinghua University in 2012 and received the Ph.D degree. She is currently a teacher in Capital Normal University. Her research interests include formal verification and the verification of robot safety.
e-mail:rwang04@cnu.edu.cn



ZHANG Qianying graduated from University of Chinese Academy of Sciences in 2015 and received the Ph.D degree. She is currently a teacher in Capital Normal University. Her research interests include real-time operating system and formal verification.
e-mail:zsjzqy@gmail.com