Asymptotic Magnitude Bode Plots of Fractional-Order Transfer Functions

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***Abstract*—Development of asymptotic magnitude Bode plots for integer-order transfer functions is a well-established topic in the control theory. However, construction of such plots for the fractional-order transfer functions has not received much attention in the existing literature. In the present paper, we investigate in this direction and derive the procedures for sketching asymptotic magnitude Bode plots for some of the popular fractional-order controllers such as PI^{α}, $[PI]^{\alpha}$, PD^{β}, $[PD]^{\beta}$, and $PI^{\alpha}D^{\beta}$. In addition, we deduce these plots for general fractional commensurate-order transfer functions as well. As applications of this work, we illustrate (i) the analysis of the designed fractional-control loop and (ii) the identification of fractional-order transfer function from a given plot.**

***Index Terms*—Asymptotic magnitude bode, commensurate-Order, fractional-Order.**

I. Introduction

Bode plot [1], [2] plays an important role in the control theory for graphically visualizing the frequency behavior of a transfer function. Generally, software tools such as MATLAB, SCILAB, etc. are used for obtaining an accurate Bode plot as it involves significant amount of computational efforts. However, one can sketch a good straight-line approximation of the exact Bode plot known as asymptotic Bode plot [3], [4] by doing a few simple calculations.

Asymptotic Bode plots are useful for quick manual analysis of a designed control system with a reasonable degree of accuracy [5]. They are also important for understanding the role of each parameter of the given transfer function in deciding the shape of its Bode response [6]. This knowledge, in particular about the controller structures is very important to a design engineer for manually tuning the control system. The procedures to sketch asymptotic Bode plots of *integer-order* transfer functions are well-established in the existing theory [3] [4].

Fractional calculus [7] generalizes the notion of *integer-order* transfer functions to arbitrary orders, which leads to the existence of *fractional-order* transfer functions [8]$-$[10]. The control theory finds application of fractional calculus in the form of fractional-order controllers such as PI^{α}, $[PI]^{\alpha}$, PD^{β}, $[PD]^{\beta}$, $PI^{\alpha}D^{\beta}$, etc. which have *fractional-order* transfer functions [9]$-$[11].

For a given fractional-order transfer function, one may consider its integer-order approximation to sketch its asymptotic Bode plot using existing procedures for integer-order transfer functions. However, the development of asymptotic Bode plots of fractional-order transfer functions in their original irrational-form has not received much attention in the literature. In [11], [12], only a brief mention is found about such plots in the context of fractional-order lead compensator. Therefore, it is important to formulate and analytically justify development of asymptotic plots for general fractional-order transfer functions, which is undoubtedly lacking in the existing literature. In the current paper, we obtain asymptotic magnitude Bode plots of: a) PI^{α}, $[PI]^{\alpha}$, PD^{β}, $[PD]^{\beta}$, $PI^{\alpha}D^{\beta}$ controllers and b) general fractional commensurate-order transfer functions. The contributions of our paper are summarized as follows:

1) To define basic fractional-order terms and develop their individual asymptotic magnitude Bode plots.
2) To utilize above plots for developing asymptotic magnitude Bode plots of:
   a) Fractional-order controllers such as PI^{α}, $[PI]^{\alpha}$, PD^{β}, $[PD]^{\beta}$, $PI^{\alpha}D^{\beta}$.
   b) General fractional commensurate-order transfer functions.
3) To illustrate the applications of these plots for:
   a) Performance analysis of designed fractional-order control loop.
   b) Identifying fractional-order transfer function from given asymptotic magnitude plot.

II. Asymptotic Magnitude Bode Plots of Basic Terms

We introduce a few basic fractional-order terms given in Table I, where K, a, a_1, $a_2 \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}_{>0}$.

First, we explain the development of asymptotic magnitude Bode plots for terms namely, constant gain, fractional zero, and fractional double-term pole. Later, such plots are obtained for the remaining terms.

A. Constant Gain

It is easy to see that for the constant gain transfer function $T(s) = K$, the magnitude $|T(j\omega)|_{dB} = 20log_{10}|K|$, $\forall\omega$. Therefore, to draw magnitude Bode plot of constant gain, one just has to sketch a horizontal line at $20log_{10}|K|$. In Bode

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plot, x-axis represents frequency (ω) in rad/s on a logarithmic scale and y-axis represents magnitude in dB on a linear scale.

TABLE I
BASIC FRACTIONAL-ORDER TERMS

TERM DESCRIPTION	TRANSFER FUNCTION ($T(s)$)
CONSTANT GAIN	K
FRACTIONAL ZERO	$s^\alpha + a$
FRACTIONAL POLE	$\frac{1}{s^\alpha + a}$
FRACTIONAL ZERO AT ORIGIN	s^α
FRACTIONAL POLE AT ORIGIN	$\frac{1}{s^\alpha}$
FRACTIONAL [ZERO]	$(s+a)^\alpha$
FRACTIONAL [POLE]	$\frac{1}{(s+a)^\alpha}$
FRACTIONAL DOUBLE-TERM ZERO	$s^{\alpha+\beta} + a_1 s^\alpha + a_2$
FRACTIONAL DOUBLE-TERM POLE	$\frac{1}{s^{\alpha+\beta} + a_1 s^\alpha + a_2}$

B. Fractional Zero

The transfer function of fractional zero is given by:

$$T(s) = (s)^\alpha + a$$

Substituting $s = j\omega$ (where, $\omega \in \mathbb{R}_{\geq 0}$) results into:

$$T(j\omega) = (j\omega)^\alpha + a$$

Therefore, the magnitude in dB is given by,

$$|T(j\omega)|_{dB} = 20log_{10}\left(a^2 + \omega^{2\alpha} + 2a\omega^\alpha cos\left(\frac{\pi\alpha}{2}\right)\right)^{\frac{1}{2}}$$

In the sum $\left(a^2 + \omega^{2\alpha} + 2a\omega^\alpha cos\left(\frac{\pi\alpha}{2}\right)\right)$, the term a^2 dominates at lower frequencies whereas the term $\omega^{2\alpha}$ dominates at higher frequencies. For the intended approximation, we choose the corner frequency (or break frequency) ω_{cr} such that these terms are equal, that is, $a^2 = \omega^{2\alpha}|_{\omega=\omega_{cr}}$. From which, one obtains the corner frequency, $\omega_{cr} = |a|^{\frac{1}{\alpha}}$. Thus, the following approximation of the magnitude is obtained:

1) For $\omega \leq \omega_{cr}$, $|T(j\omega)|_{dB} = 20log_{10}\left(a^2\right)^{\frac{1}{2}} = 20log_{10}|a|$.
2) For $\omega > \omega_{cr}$, $|T(j\omega)|_{dB} = 20log_{10}\left(\omega^{2\alpha}\right)^{\frac{1}{2}} = 20\alpha log_{10}\omega$.

Based on the above discussion, we lay down the following procedure to construct the asymptotic magnitude plot for $(s^\alpha + a)$ shown in Fig. 1:

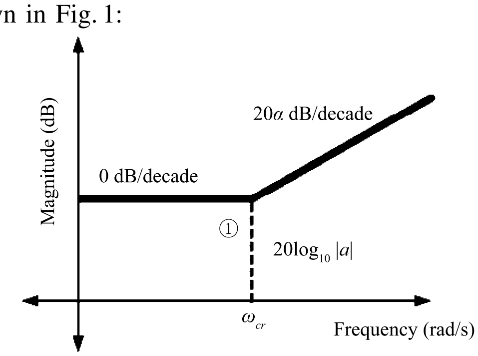


Fig. 1. Asymptotic Magnitude Bode Plot for Fractional Zero.

Procedure:

1) Compute the corner frequency $\omega_{cr} = |a|^{\frac{1}{\alpha}}$ and locate point ① at magnitude $20log_{10}|a|$.
2) Draw a line with slope 0 dB/decade for $\omega \leq \omega_{cr}$, and a line with slope 20α dB/decade for $\omega > \omega_{cr}$ as shown in Fig. 1.

Comparison with Real Magnitude Bode Plot:

1) At this point, let us compare the asymptotic magnitude Bode plot with the real magnitude Bode plot for the *Fractional Zero* term $(s^\alpha + a)$. For this purpose, let the numerical values be: $\alpha = 0.9, a = 2$.[1]
2) Fig. 2 shows the real as well as asymptotic magnitude Bode plots for the fractional zero term, $(s^{0.9} + 2)$.
3) As seen from Fig. 2, the asymptotic plot follows the real plot quite closely, thereby confirming the correctness of our asymptotic formulation.

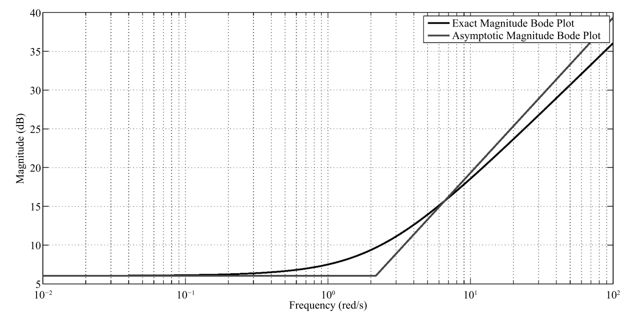


Fig. 2. Real and Asymptotic Magnitude Bode Plot for $(s^{0.9} + 2)$

C. Fractional Double-Term Pole

The transfer function of fractional double-term pole is given by $T(s) = \frac{1}{(s^{\alpha+\beta} + a_1 s^\alpha + a_2)}$. Substituting $s = j\omega$ (where, $\omega \in \mathbb{R}_{\geq 0}$) leads to:

$$T(j\omega) = \frac{1}{(j\omega)^{\alpha+\beta} + a_1(j\omega)^\alpha + a_2}$$

The decibel magnitude of $T(j\omega)$ is given by,

$$|T(j\omega)|_{dB} = -20log_{10}\Bigg(\omega^{2(\alpha+\beta)} + a_1^2\omega^\alpha + a_2^2 + 2a_1\omega^{2\alpha+\beta}cos\left(\frac{\pi\beta}{2}\right) + 2a_2\omega^{\alpha+\beta}cos\left(\frac{\pi(\alpha+\beta)}{2}\right) + 2a_1a_2\omega^\alpha cos\left(\frac{\pi\alpha}{2}\right)\Bigg)^{\frac{1}{2}}$$

In the sum $\left(\omega^{2(\alpha+\beta)} + a_1^2\omega^\alpha + a_2^2 + 2a_1\omega^{2\alpha+\beta}cos\left(\frac{\pi\beta}{2}\right) + 2a_2\omega^{\alpha+\beta}cos\left(\frac{\pi(\alpha+\beta)}{2}\right) + 2a_1a_2\omega^\alpha cos\left(\frac{\pi\alpha}{2}\right)\right)$, the term a_2^2 dominates at lower frequencies whereas the term $\omega^{2(\alpha+\beta)}$ dominates at higher frequencies. For the approximation purpose, the corner frequency ω_{cr} is chosen such that the dominant terms are equal,

$$a_2^2 = [\omega^{2(\alpha+\beta)}]_{\omega=\omega_{cr}}$$

Therefore, one gets the corner frequency,

$$\omega_{cr} = |a_2|^{\frac{1}{(\alpha+\beta)}}$$

Hence, the following magnitude approximation is obtained:
1) For $\omega \leq \omega_{cr}$:

$$|T(j\omega)|_{dB} = -20log_{10}|a_2|$$

[1]Although only one numerical example has been illustrated here, one may consider different sets of parameter values for the numerical confirmation.

2) For $\omega > \omega_{cr}$:

$$|T(j\omega)|_{dB} = -20log_{10}|\omega^{(\alpha+\beta)}| = -20(\alpha+\beta)log_{10}\omega$$

From the discussion above, following procedure is stated to sketch asymptotic magnitude Bode plot for fractional double-term pole $\frac{1}{(s^{\alpha+\beta}+a_1 s^{\alpha}+a_2)}$ shown in Fig. 3:

Procedure:

1) Compute the corner frequency $\omega_{cr} = |a_2|^{\frac{1}{(\alpha+\beta)}}$ and locate point ① at magnitude $-20log_{10}|a_2|$.

2) Draw a line with slope 0 dB/decade for $\omega \leq \omega_{cr}$, and a line with slope $-20(\alpha+\beta)$ dB/decade for $\omega > \omega_{cr}$ as shown in Fig. 3.

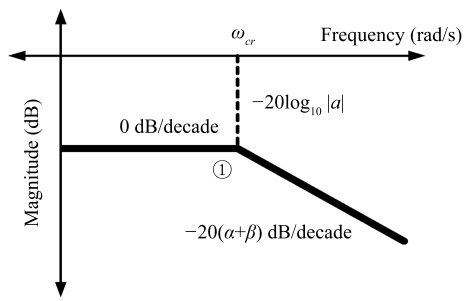


Fig. 3. Asymptotic Magnitude Bode Plot for Fractional double-term Pole.

Comparison with Real Magnitude Bode Plot:

1) Let us consider a numerical example of finding the magnitude Bode plot of the *Fractional Double-Term Pole* term, with the parameters: $\alpha = 0.5, \beta = 0.9, a_1 = 2, a_2 = 3$.

2) Fig. 4 shows the real as well as asymptotic magnitude Bode plots for fractional double-term pole, $\frac{1}{(s^{(0.5+0.9)}+2s^{0.5}+3)}$.

As seen from Fig. 4, the asymptotic plot follows the real plot quite closely, thereby confirming the correctness of our formulation.

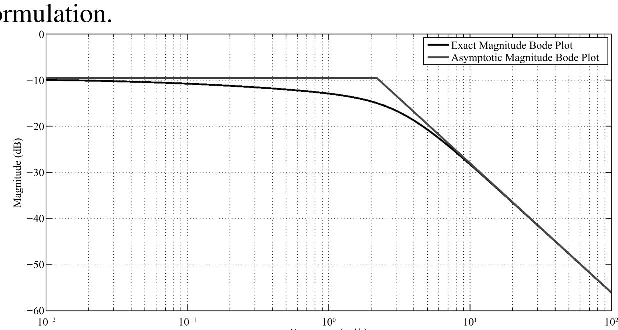


Fig. 4. Real and Asymptotic Magnitude Bode Plot for $\frac{1}{(s^{(0.5+0.9)}+2s^{0.5}+3)}$

Similarly, one can obtain such plots for terms: $\frac{1}{s^{\alpha}+a}$, s^{α}, $\frac{1}{s^{\alpha}}$, $(s+a)^{\alpha}$, $\frac{1}{(s+a)^{\alpha}}$, and $s^{\alpha+\beta}+a_1 s^{\alpha}+a_2$. The results are summarized in Tables II and III. Also, in each case, the comparison is made between the real and asymptotic plots using suitable examples. It can be seen from the Tables II and III that for the numerical cases under consideration, the asymptotic and real Bode plots are quite close to each other.[2]

[2]As seen from Table II, in case of fractional zero at origin (s^{α}) the asymptotic and real plots 'coincide' (as the magnitude with both, the real and asymptotic formulation, turns out to be: $20\alpha log_{10}w$). Therefore, for the numerical example, the plots overlap each other. The same is also true in case of fractional pole at origin ($\frac{1}{s^{\alpha}}$).

Remark 1: It can be observed in Tables II and III that since the transfer functions of fractional zero and fractional pole are reciprocal to each other, their magnitude plots are mirror images of each other with respect to ω-axis. This is also true for pairs such as fractional pole and zero at origin, fractional [pole] and [zero], fractional double-term pole and zero.

D. Asymptotic Magnitude Bode Plots for Fractional-Order Controllers

In the present subsection, the asymptotic magnitude Bode plots of basic fractional-order terms are used to obtain such plots for fractional-order controllers, PI^{α}, $[PI]^{\alpha}$, PD^{β}, $[PD]^{\beta}$, and $PI^{\alpha}D^{\beta}$.

Let us consider PI^{α} controller which has the transfer function:

$$C(s) = K_p\left(1+\frac{K_i}{s^{\alpha}}\right) = (K_p)\left(s^{\alpha}+K_i\right)\left(\frac{1}{s^{\alpha}}\right) \quad (1)$$

As observed in (1), PI^{α} is expressed as a product of transfer functions of constant gain, fractional zero and fractional pole at origin. Therefore, the asymptotic magnitude Bode plot of PI^{α} can be obtained by *adding* such plots of its constituent elements as shown in Table IV. Similarly, one can develop the asymptotic magnitude Bode plots for $[PI]^{\alpha}$, PD^{β}, $[PD]^{\beta}$, and $PI^{\alpha}D^{\beta}$ controllers as summarized in Table V.

Remark 2: The asymptotic magnitude Bode plots of fractional-order controllers are obtained from those of basic fractional-order terms. Therefore, we do not pursue the comparison between asymptotic and real magnitude Bode plots for the fractional-order controllers exclusively. Nevertheless, in Section IV, a numerical example is considered to demonstrate the analysis of fractional control loop using the asymptotic formulation, wherein the real and asymptotic plots have been compared.

III. Asymptotic Magnitude Bode Plots for General Fractional Commensurate-Order Transfer Functions

Let us consider a general fractional-order transfer function,

$$\frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m}+b_{m-1}s^{\beta_{m-1}}+...+b_0 s^{\beta_0}}{a_n s^{\alpha_n}+a_{n-1}s^{\alpha_{n-1}}+...+a_0 s^{\alpha_0}} \quad (2)$$

The transfer function (2) represents a *commensurate*-order system, if there exists a greatest common divisor $q \in \mathbb{R}$ such that $\alpha_i = qe_i(i = 0,1,2,\ldots,n), \beta_k = qf_k(k = 0,1,2,\ldots,m); e_i, f_k \in \mathbb{Z}$. Here, q is called the commensurate order, which can be rational or irrational. Therefore,

$$T(s) := \frac{Y(s)}{U(s)} = \frac{P(s^q)}{Q(s^q)}$$

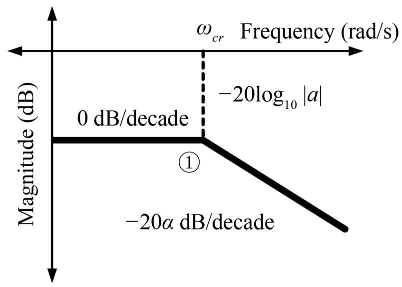
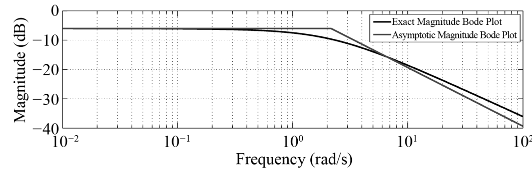
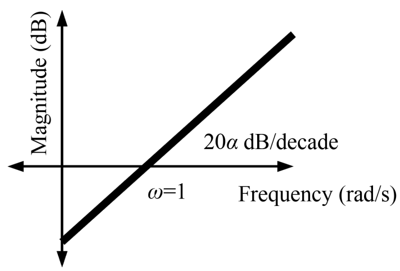
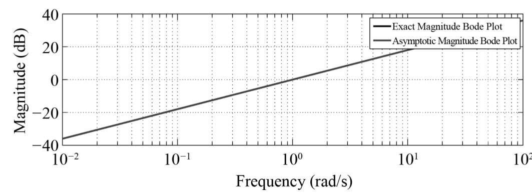
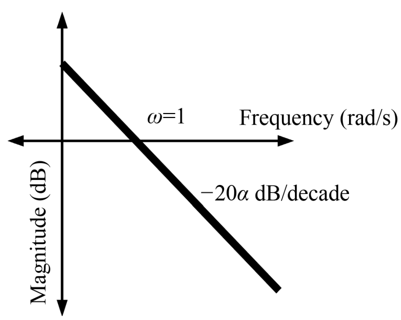
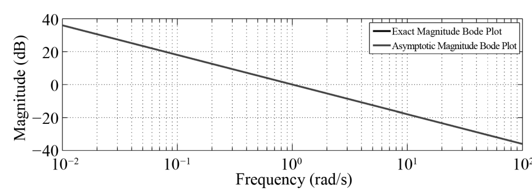
where, $P(.), Q(.)$ are polynomial functions. If $p = s^q$, then,

$$T(p) = \frac{P(p)}{Q(p)} \quad (3)$$

On factorization, (3) can be expressed as follows[3]:

[3]This is because any polynomial with real coefficients has either real roots or complex roots in pairs. The real roots lead to terms of the form $(p+c_i), (p+g_k)$ and complex roots in pairs lead to terms such as $(d_j p^2+e_j p+f_j), (h_l p^2+o_l p+q_l)$.

TABLE II
ASYMPTOTIC MAGNITUDE BODE PLOTS FOR REMAINING BASIC FRACTIONAL-ORDER TERMS

Term & Corner Frequency	**Asymptotic Magnitude Bode Plot**	**Numerical Example**
Fractional Pole, $\frac{1}{s^{\alpha}+a}$ $\omega_{cr}=\lvert a\rvert^{\frac{1}{\alpha}}$ (For Numerical Example, α=0.9,a=2)		
Fractional Zero at Origin,s^{α} (For Numerical Example, α=0.9)		
Fractional Zero at Origin, $\frac{1}{s^{\alpha}}$ (For Numerical Example, α=0.9)		

$$T(p)=\frac{\prod_{i=0}^{m_1}(p+c_i)\prod_{j=0}^{m_2}(d_jp^2+e_jp+f_j)}{\prod_{k=0}^{m_3}(p+g_k)\prod_{l=0}^{m_4}(h_lp^2+o_lp+z_l)}$$

where c_i $(i=0,1,\dots m_1)$, d_j,e_j,f_j $(j=0,1,\dots m_2)$, g_k $(k=0,1,\dots m_3)$, h_l,o_l,z_l $(l=0,1,\dots m_4)$ are real constants. m_1,m_2,m_3,m_4 are positive integers.

Now, by re-substituting $p=s^q$, one gets,

$$T(s)=\frac{\prod_{i=0}^{l_1}(s^q+c_i)\prod_{j=0}^{l_2}(d_js^{2q}+e_js^q+f_j)}{\prod_{k=0}^{l_3}(s^q+g_i)\prod_{l=0}^{l_4}(h_ls^{2q}+o_ls^q+z_l)} \quad (4)$$

It is seen that (4) is composed of fractional zeros, fractional poles, fractional double-term zeros and fractional double-term poles. Hence, one can construct asymptotic Bode plots of $T(s)$ by adding such plots of their constituent terms similar to the PI^{α} case explained in Table IV.

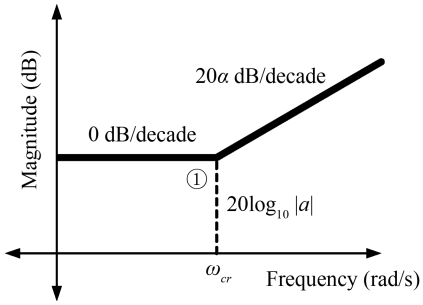
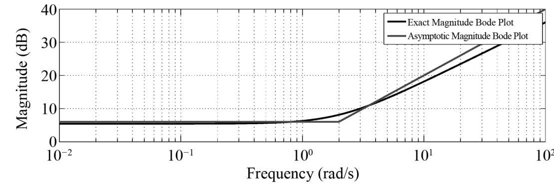
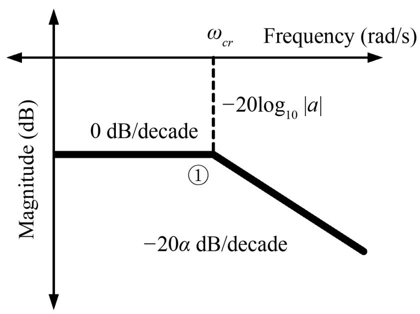
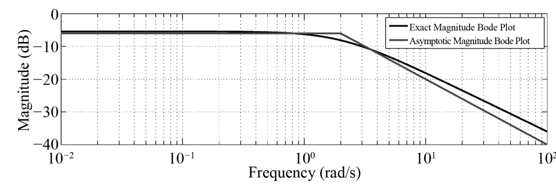
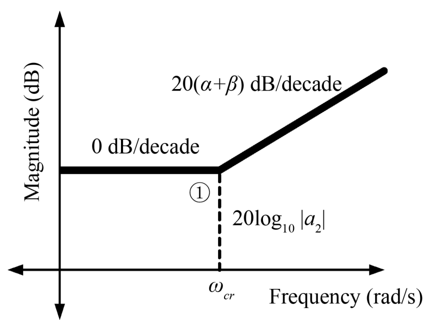
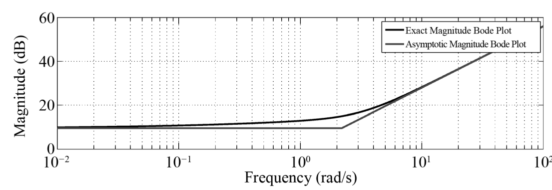
IV. APPLICATIONS OF ASYMPTOTIC MAGNITUDE BODE PLOTS

In this section, we demonstrate two applications of asymptotic magnitude Bode formulations, 1) Analysis of fractional control loop and 2) Identification of fractional-order transfer function from asymptotic magnitude plot.

A. Analysis of Fractional Control Loop

Let us suppose that we have tuned a $[PD]^{\beta}$ controller for a type-1 motion plant of the form $\frac{K}{s(Ts+1)}$ to meet required gain crossover frequency (ω_{gc}), phase margin (ϕ_m), and isodamping property [13] by following the methodology given in [14]. The numerical values are: $K=1,T=0.4,\omega_{gc}=10$ $rad/s,\phi_m=70°$. The plant $(G(s))$ and designed controller $(C(s))$ are as follows:

TABLE III
ASYMPTOTIC MAGNITUDE BODE PLOTS FOR REMAINING BASIC FRACTIONAL-ORDER TERMS

Term & Corner Frequency	Asymptotic Magnitude Bode Plot	Numerical Example
Fractional [Zero], $(s+a)^{\alpha}$ $\omega_{cr}=\|a\|$ (For Numerical Example, $\alpha=0.9, a=2$)		
Fractional [Pole], $\frac{1}{(s+a)^{\alpha}}$ $\omega_{cr}=\|a\|$ (For Numerical Example, $\alpha=0.9, a=2$)		
Fractional Double-Term Zero, $(s^{\alpha+\beta}+a_1 s^{\alpha}+a_2)$ $\omega_{cr}=\|a_2\|^{\frac{1}{(\alpha+\beta)}}$ (For Numerical Example, $\alpha=0.9, \beta=0.9, a_1=2, a_2=3$)		

$$
\begin{aligned}
G(s) &= \tfrac{1}{s(0.4s+1)}, C(s) = 16.7780(1+0.2992s)^{0.7826} \\
\therefore L(s) & \\
&= C(s)G(s) = 16.7780(1+0.2992s)^{0.7826}\tfrac{1}{s(0.4s+1)} \\
&= (16.3143)\left(s+\tfrac{1}{0.2992}\right)^{0.7826}\left(\tfrac{1}{s^1}\right)\tfrac{1}{(s^1+2.5)}
\end{aligned} \tag{5}
$$

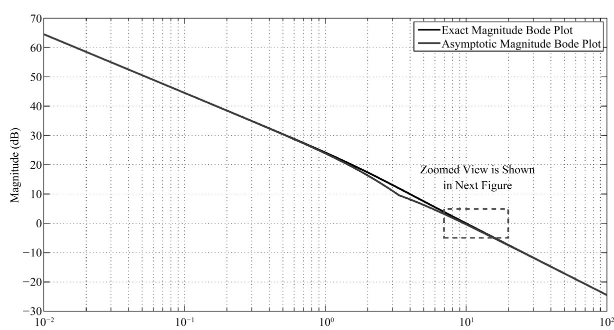


Fig. 5. Application of Asymptotic Magnitude Plot for Loop Analysis.

Our focus is to illustrate the usefulness of earlier formulations for analyzing magnitude Bode characteristics of the designed loop transfer function $L(s)$. More precisely, we intend to verify the gain crossover frequency met by $L(s)$. From (5), it can be seen that $L(s)$ is composed of basic terms defined in Section II. One can draw their individual asymptotic plots and add them to get the plot for $L(s)$. Fig. 5 presents exact and asymptotic magnitude bode plots for $L(s)$.

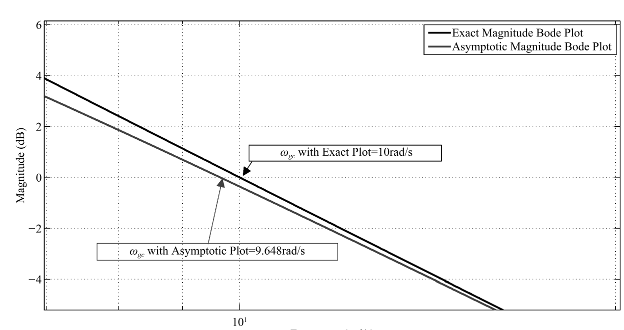


Fig. 6. Identification of Fractional-Order transfer function from Asymptotic Magnitude Bode Plot

TABLE IV
ASYMPTOTIC MAGNITUDE BODE PLOT FOR PI^{α} CONTROLLER

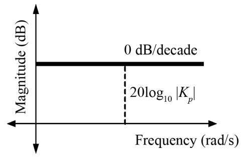
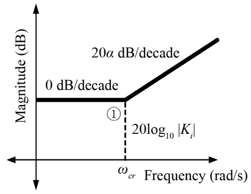
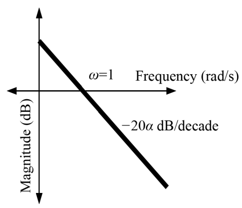
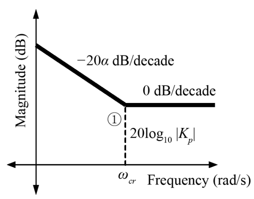
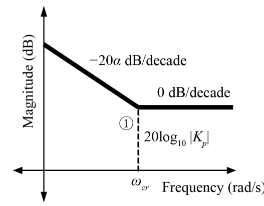
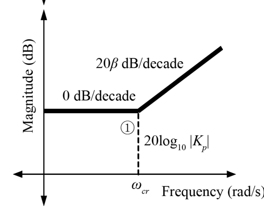
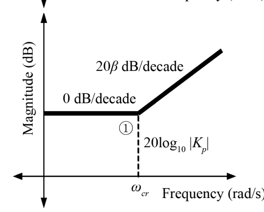
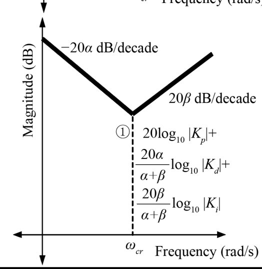
Term	Asymptotic Magnitude Bode Plot
Constant Gain, K_p	
Fractional Zero, $s^{\alpha}+K_i$ $\omega_{cr}=\|K_i\|^{\frac{1}{\alpha}}$	
Fractional Pole at Origin, $\frac{1}{s^{\alpha}}$	
On adding the above plots, one gets:	
PI^{α} Controller $=K_p(1+\frac{K_i}{s^{\alpha}})$ $=(K_p)(s^{\alpha}+K_i)(\frac{1}{s^{\alpha}})$ $\omega_{cr}=\|K_p\|^{\frac{1}{\alpha}}$	

TABLE V
ASYMPTOTIC MAGNITUDE BODE PLOTS FOR OTHER FRACTIONAL-ORDER CONTROLLERS

Controller	Asymptotic Magnitude Bode Plot
$[PI]^{\alpha}$ $=K_p(1+\frac{K_i}{s})^{\alpha}$ $=K_p(s+K_i)^{\alpha}(\frac{1}{s^{\alpha}})$ $\omega_{cr}=\|K_i\|$	
PD^{β} $=K_p(1+K_d s^{\beta})$ $=K_pK_d(s^{\beta}+\frac{1}{K_d})$ $\omega_{cr}=\|\frac{1}{K_d}\|^{\frac{1}{\beta}}$	
PD^{β} $=K_p(1+K_d s)^{\beta}$ $=K_pK_d^{\beta}(s+\frac{1}{K_d})^{\beta}$ $\omega_{cr}=\|\frac{1}{K_d}\|$	
$PI^{\alpha}D^{\beta}$ $=K_p(1+\frac{K_i}{s^{\alpha}}+K_d s^{\beta})$ $=K_pK_d(s^{\alpha+\beta}+\frac{s^{\alpha}}{K_d}+\frac{K_i}{K_d})\frac{1}{s^{\alpha}}$ $\omega_{cr}=\|\frac{K_i}{K_d}\|^{\frac{1}{(\alpha+\beta)}}$	

The zoomed view of a selected portion of Fig. 5 is shown in Fig. 6. From Fig. 6, it is seen that the ω_{gc} values with asymptotic and exact magnitude Bode plots (9.648 and 10 rad/s, respectively) are quite close to each other. This confirms the correctness of our formulations.

B. Identification of Fractional-Order Transfer Function

Let us consider a general asymptotic magnitude Bode plot as shown in Fig. 7 (Where, $a_1, a_2, \ldots a_n \in \mathbb{R}_{>0}, b_1, b_2, \ldots b_{n+1} \in \mathbb{R}$). It is desired to identify the fractional-order transfer function corresponding to the asymptotic magnitude Bode plot in Fig. 7. It must be noted that in Fig. 7, the straight-line approximations assume any arbitrary slope. (In integer-order transfer function case, such slopes are always integer multiples of 20.)

Prior to identification, it is essential to consider the asymptotic magnitude Bode plots of following composite terms:

1) ks^{α} (where, $k, \alpha \in \mathbb{R}$)

ks^{α} is composed of constant gain k and the term s^{α}. Based on the value of α, there are following possible cases:

a) $\alpha > 0$: In such case, s^{α} is fractional zero at origin. Asymptotic magnitude Bode plot of ks^{α} obtained from its constituent terms is shown in Fig. 8. Figure is sketched for $|k| > 1$. One can also sketch the corresponding one for $|k| < 1$.

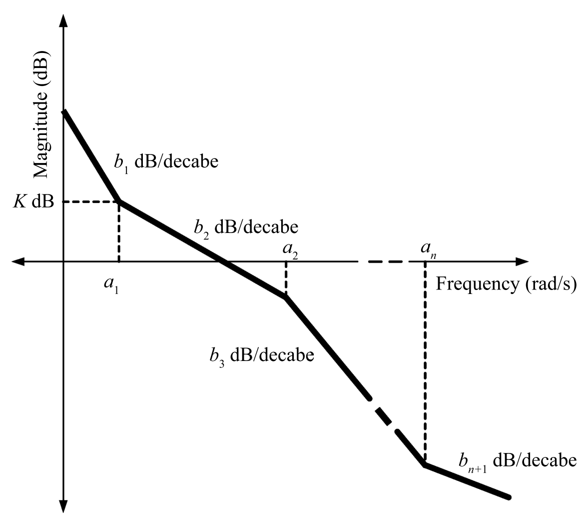


Fig. 7. Identification of Fractional-Order transfer function from Asymptotic Magnitude Bode Plot

b) $\alpha < 0$: In this case, s^{α} represents fractional pole at origin. Fig. 9 shows the asymptotic magnitude Bode plot for ks^{α}.

c) $\alpha = 0$: For this case, ks^{α} reduces to k. The discussion for asymptotic magnitude plot for such a term was made in Section II-A.

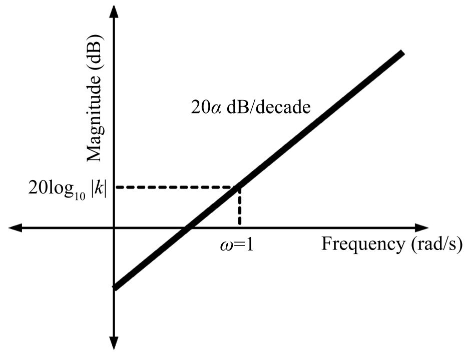


Fig. 8. Asymptotic Magnitude Bode Plot for ks^{α} when $\alpha < 0$.

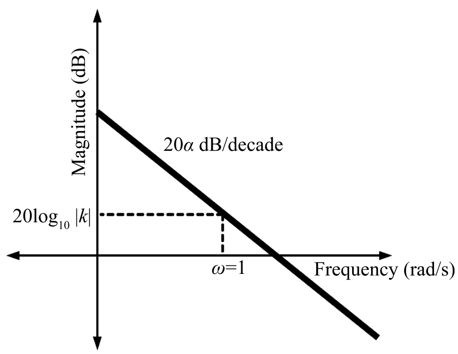


Fig. 9. Asymptotic Magnitude Bode Plot for $\frac{(s+a)^{\alpha}}{a^{\alpha}}$ when $\alpha > 0$.

Remark: It can be inferred from the above asymptotic plots that the term ks^{α} is identified when a line of given slope offset by a known magnitude is observed.

2) $\frac{(s+a)^{\alpha}}{a^{\alpha}}$ (where, $a \in \mathbb{R}$, $\alpha \in R_{\neq 0}$)

$\frac{(s+a)^{\alpha}}{a^{\alpha}}$ is composed of constant gain $\frac{1}{a^{\alpha}}$ and the term $(s+a)^{\alpha}$. When $\alpha > 0$, $(s+a)^{\alpha}$ represents fractional [zero]. For such case, the asymptotic magnitude plot of $\frac{(s+a)^{\alpha}}{a^{\alpha}}$ obtained from its constituent elements is as shown in Fig. 10.

On the other hand, for $\alpha < 0$, $(s+a)^{\alpha}$ represents fractional [pole]. Therefore, the asymptotic magnitude plot of $\frac{(s+a)^{\alpha}}{a^{\alpha}}$ in this case takes the shape as shown in Fig. 11.

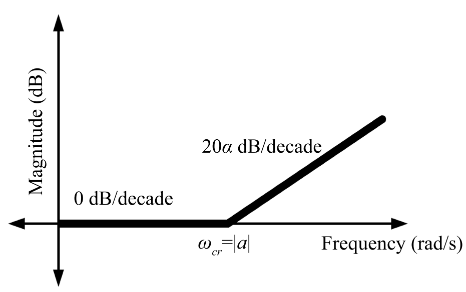


Fig. 10. Asymptotic Magnitude Bode Plot for $\frac{(s+a)^{\alpha}}{a^{\alpha}}$ when $\alpha > 0$.

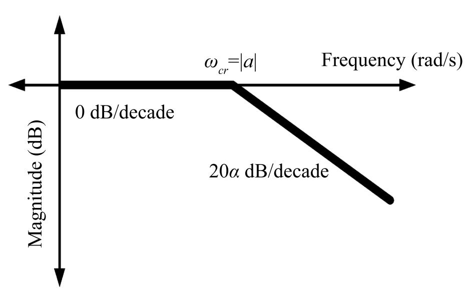


Fig. 10. Asymptotic Magnitude Bode Plot for $\frac{(s+a)^{\alpha}}{a^{\alpha}}$ when $\alpha < 0$.

Remark: It can be observed from Figs. 10 and 11 that,

1) $\frac{(s+a)^{\alpha}}{a^{\alpha}}$ with $\alpha > 0$ is identified when there is an 'increase' in slope at given corner frequency.

2) $\frac{(s+a)^{\alpha}}{a^{\alpha}}$ with $\alpha < 0$ is identified when there is a 'decrease' in slope at given corner frequency.

Based on above discussion, we identify the fractional-order transfer function from the asymptotic magnitude Bode plot given in Fig. 7 as follows:

1) From Fig. 7, it is seen that for the frequency range from 0 to a_1, the plot is a line with slope b_1 dB/decade. Recalling Remark 3, we identify the corresponding term as ks^{α} with $\alpha = \frac{b_1}{20}$ (Since, $20\alpha = b_1$). The constant k is obtained as follows:

From Fig. 7, $20log_{10}|ks^{\alpha}|_{s=ja_1} = K$.

Therefore, $|k| = \frac{10^{\frac{K}{20}}}{a_1^{\alpha}} \implies k = \pm\frac{10^{\frac{K}{20}}}{a_1^{\frac{b_1}{20}}}$.

2) At corner frequency a_1, there is an observed increase of slope from b_1 to b_2. From Remark 4, this corresponds to the term $\frac{(s+a)^{\alpha}}{a^{\alpha}}$ with $\alpha = \frac{b_2-b_1}{20}$, $a = a_1$ (since, $b_2 > b_1$, $\alpha > 0$). Similarly, at corner frequency a_2, there is an observed decrease of slope from b_2 to b_3. From Remark 4, we get the corresponding term as $\frac{(s+a)^{\alpha}}{a^{\alpha}}$ with $\alpha = \frac{b_3-b_2}{20}$, $a = a_2$ (since, $b_3 < b_2$, $\alpha < 0$). One can similarly obtain the terms for observed change in slopes at a_3, a_4, $\ldots$, a_n.

3) The individual identified terms are multiplied to get the complete transfer function $T(s)$ for the asymptotic magnitude plot given in Fig. 7 as follows:

$$T(s) = \pm\left(\frac{10^{\frac{K}{20}} s^{\frac{b_1}{20}}}{a_1^{\frac{b_1}{20}}}\right)\left(\frac{(s+a_1)^{\frac{b_2-b_1}{20}}}{a_1^{\frac{b_2-b_1}{20}}}\right)\left(\frac{(s+a_2)^{\frac{b_3-b_2}{20}}}{a_2^{\frac{b_3-b_2}{20}}}\right)\cdots\left(\frac{(s+a_n)^{\frac{b_{n+1}-b_n}{20}}}{a_n^{\frac{b_{n+1}-b_n}{20}}}\right)$$

It is important to note that the above general case considers asymptotic magnitude Bode plots containing lines with arbitrary slopes. Therefore, it also includes the integer-order transfer function cases when slopes are integer multiples of 20.

V. Conclusion

In this paper, we presented the construction of asymptotic magnitude Bode plots for popular fractional-order controllers such as PI^{α}, $[PI]^{\alpha}$, PD^{β}, $[PD]^{\beta}$, and $PI^{\alpha}D^{\beta}$. The plots were also developed for general fractional commensurate-order transfer functions. The applicability of this work to quickly analyze the performance of designed control system containing fractional-order elements was demonstrated with the help of a numerical example. We also showed the usefulness of our formulation in identifying fractional-order transfer function from the given asymptotic magnitude Bode plot. Development of asymptotic phase Bode plots for fractional-order transfer functions can be an another interesting direction to this work.

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