An Implementation of Haar Wavelet Based Method for Numerical Treatment of Time-fractional Schrödinger and Coupled Schrödinger Systems

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***Abstract*—The objective of this paper is to solve the time-fractional Schrödinger and coupled Schrödinger differential equations (TFSE) with appropriate initial conditions by using the Haar wavelet approximation. For the most part, this endeavor is made to enlarge the pertinence of the Haar wavelet method to solve a coupled system of time-fractional partial differential equations. As a general rule, piecewise constant approximation of a function at different resolutions is presentational characteristic of Haar wavelet method through which it converts the differential equation into the Sylvester equation that can be further simplified easily. Study of the (TFSE) is theoretical and experimental research and it also helps in the development of automation science, physics, and engineering as well. Illustratively, several test problems are discussed to draw an effective conclusion, supported by the graphical and tabulated results of included examples, to reveal the proficiency and adaptability of the method.**

***Index Terms*—Fractional calculus, haar wavelets, operational matrix, wavelets.**

I. Introduction

IN recent decades, fractional calculus (calculus of integrals and derivatives of any arbitrary real order) has attained appreciable fame and importance due to its manifest uses in apparently diverse and outspread fields of science. Certainly, it provides potentially helpful tools for solving integral and differential equations and many other problems of mathematical physics. The fractional differential equations have become crucial research field essentially due to their immense range of utilization in engineering, fluid mechanics, physics, chemistry, biology, viscoelasticity etc. Numerous mathematicians and physicists have been studying the properties of fractional calculus [1], [2] and have established several methods for accurate analytical and numerical solutions of fractional differential equations, such as the variational iteration method [3], differential transform method [4], homotopy analysis method [5], Jacobi spectral tau and collocation method [6]−[8], Laplace transform method [9], homotopy perturbation method [10], Adomian decomposition method [8], [9], high-order finite element methods [13] and many others [14], [19]. The scope and distinct aspects of fractional calculus have been written by many authors in Refs [20]−[22].

In the past few years, there has been an extensive attraction in employing the spectral method (see [23]−[25]) for numerically solving the copious type of differential and integral equations. The spectral methods have an exponential quota of convergence and high level of efficiency. Spectral methods are to express the approximate solution of the problem in term of a finite sum of certain basis functions and then selection of coefficients in order to reduce the difference between the exact and approximate solutions as much as possible. The spectral collocation method is a distinct type of spectral methods, that is more relevant and extensively used to solve most of differential equations [26].

For the reason of the distinctive attributes of wavelet theory in representing continuous functions in the form of discontinuous functions [27], its applications as a mathematical tool is widely expanding nowadays. Besides image processing and signal decomposition it is also used to assess many other mathematical problems, such as differential and integral equations. Wavelets comprise the incremental conception between two consecutive levels of resolution, called multi-resolution. The first component of multi-resolution analysis is vector spaces. For each vector space, another vector space of higher resolution is found and this continues until the final image or signal is executed. The basis of each of these vector spaces acts as the scaling function for the wavelets. Each vector space having an orthogonal component and a basis function is said to be the wavelet [28].

Up till now, a number of wavelet families have been presented by different authors, but among all Haar wavelet are considered to be the easiest wavelets family. Haar wavelet was introduced in 1910 by Hungarian mathematician Alfred Haar. These wavelets are obtained from Daubechies wavelets of order 1, which consist of piecewise constant functions on the real axis that can take only three values, -1, 0 and 1. Here we are using collocation method, by increasing the level of resolution, collocation points are also increasing and level of accuracy too. Haar wavelet collocation method is extensively used due to its constructive ability of being smooth, fast, convenient and being computationally attractive [29]. In addition, it has the competency to reduce the computations for solving differential equations by converting them into some system of algebraic equations. The main advantages of

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the proposed algorithm are, its simple application and no residual or product operational matrix is required. The method is well addressed in [22], [29]−[32].

The time-fractional Schrödinger equation (T-FSE) differs from the standard Schrödinger equation. The first-order time derivative is replaced by a fractional derivative, it makes the problem overall in time. It describes, how the quantum state (physical situation) of a quantum system changes with time, soliton dispersion, deep water waves, molecular orbital theory and the potential energy of a hydrogen-like atom (fractional 'Bohr atom'). The aim of this work is to explore the numerical solutions of the time-fractional Schrödinger equations by using Haar wavelet method. Due to the large number of applications of the Schrödinger equation in different aspects of quantum mechanics and engineering, many attempts have been exercised on analytical and numerical methods to calculate the approximate solution of (T-FSE). Some of them are studied [6], [10], [18], [33]−[36], and enumerated here for better perception of the presented analysis. Also, the existence and uniqueness of solutions of fractional Schrödinger equation have been proved by multiple authors [35], [37], [38].

II. Preliminaries

In this section, some notations and properties of fractional calculus, the basis of Haar function approximation for partial differential equation and solution of Haar by multi-resolution analysis are given that will help us in exploring the main theme of the paper.

A. Riemann-Liouville Differential and Integral Operator

Assume $\nu > 0$, $m = \lceil \nu \rceil$ and $f(x,t) \in C^m([0,1] \times [0,1])$ then the partial Caputo fractional derivative of $f(x,t)$ with respect to t is defined as

$$\frac{\partial^\nu}{\partial t^\nu} f(x,t) = \begin{cases} I_t^{m-\nu} \frac{\partial^m}{\partial t^m} f(x,t) \\ \frac{\partial^m}{\partial t^m} f(x,t) \end{cases} \tag{1}$$

where I_t^ν is the Riemann-Liouville fractional integral, given as

$$\begin{aligned} I_t^\nu f(t) &= \frac{1}{\Gamma(\nu)} \int_0^t (t-\varphi)^{\nu-1} f(\varphi) d\varphi \\ I_t^0 f(t) &= f(t) \end{aligned} \tag{2}$$

we use the notation D_t^ν in replacement of $\frac{\partial^\nu}{\partial t^\nu}$for the Caputo fractional derivative. The Caputo fractional derivative of order $\nu > 0$ for $f(t) = t^\alpha$ is given as

$$D_t^\nu f(t) = \begin{cases} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\nu+1)} t^{\alpha-\nu}, m > \alpha \geq m-1, \\ 0 \text{ if } \alpha \in \{0,1,2,...,m-1\} \end{cases} \tag{3}$$

In the following, some main computational properties and relations of fractional integral and differential operators are defined as

$$\begin{aligned} & i) I_t^\alpha I_t^\beta f(t) = I_t^{\alpha+\beta} f(t) = I_t^\beta I_t^\alpha f(t) \\ & ii) \frac{\partial^\beta}{\partial t^\beta} I_t^\alpha f(x,t) = I_t^{\alpha-\beta} f(x,t) \\ & iii) \frac{\partial^\alpha}{\partial t^\alpha} f(x,t) = f(x,t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \frac{\partial^k f(x,t) \mid_{t=0}}{\partial t^k} \\ & = f(x,t) + \sum_{k=0}^{n-1} \zeta_k(x) t^k \end{aligned} \tag{4}$$

where,$\zeta_k(x) = -\frac{1}{k!} \frac{\partial^k f(x,t)|_{t=0}}{\partial t^k}$. For more details see [1].

B. Haar wavelets and function approximation

Basis of Haar wavelets is obtained with a multi-resolution of piecewise constant functions. Let the interval $x \in [0,1)$ be divided into $2m$ subintervals of equal length, where $m = 2^j$ and J is maximal level of resolution. Next, two parameters are introduced, $j = 0,1,2,\ldots,J$ and $k = 0,1,2, ldots, m-1$, such that the wavelet number i satisfies the relation $i = k+m+1$. The ith Haar wavelet can be determined as

$$h_i(x) = \begin{cases} 1, x \in [\vartheta_1, \vartheta_2) \\ -1, x \in [\vartheta_2, \vartheta_3) \\ 0, \text{ elsewhere} \end{cases} \tag{5}$$

where $\vartheta_1 = \frac{k}{m}, \vartheta_2 = \frac{k+0.5}{m}, \vartheta_3 = \frac{k+1}{m}$

For the case $i = 1$, corresponding scaling function can be defined as:

$$h_1 = \begin{cases} 1, x \in [\vartheta_1, \vartheta_3) \\ 0, \text{ elsewhere} \end{cases} \tag{6}$$

Here, we consider the wavelet-collocation method, therefore collocation points are generated by using,

$$x_l = \frac{l-0.5}{2m}, \ l = 1,2,3, ldots, 2m \tag{7}$$

The Haar system forms an orthonormal basis for the Hilbert space $f(t) \in L_2(0,1)$. We may consider the inner product expansion of $f(t) \in L_2([0,1))$ in Haar series [31] as:

$$f(t) \approx \langle f, \varphi \rangle \varphi(t) + \sum_{j=0}^{J-1} \sum_{i=0}^{2^j-1} \langle f, h_{j,k} \rangle h_{j,k}(t) = C^T H(t) \tag{8}$$

where, C is 1×2^J coefficient vector and $H(t) = [h_0(t), h_1(t), ldots, h_{m-1}(t)]^T$. Also, a function of two variables can be expanded by Haar wavelets [32] as:

$$u(x,t) \approx \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} u_{i,j} h_i(x) h_j(t) = H^T(x).U.H(t) \tag{9}$$

where, U is $2^J \times 2^J$ coefficient matrix calculated by the inner product $u_{i,j} = \langle h_i(x), \langle u(x,t), h_j \rangle \rangle$ The operational matrix of fractional integration of Haar function is needed to solve PDE of fractional order. A more rigorous derivation for the generalized block pulse operational matrices is proposed in [39]. The block pulse function forms a complete set of orthogonal functions which is defined in interval $[a, b)$ as

$$\psi_i(t) = \begin{cases} 1, & \frac{i-1}{m} b \le t < \frac{i}{m} b \\ 0 & \text{elsewhere} \end{cases} \tag{10}$$

for $i = 1, 2, ldots, m$ It is known that any absolutely integrable function $f(t)$ on $[a, b)$, can be expanded in block pulse functions as

$$f(t) \cong F^T \psi_{(m)}(t) \tag{11}$$

so that the mean square error of approximation is minimized. Here, $F^T = [f_1, f_2, f_3, ldots, f_m]$ and $\psi^T_{(m)}(t) = [\psi_1(t), \psi_2(t), \psi_3(t), ldots, \psi_m(t)]$. where,

$$f_i = \frac{m}{b}\int_a^b f(t)\psi_i(t)dt = \frac{m}{b}\int_{(i-1)b/m}^{(i/m)b} f(t)\psi_i(t)dt \tag{12}$$

The Riemann-Liouville fractional integral is simplified and expanded in block pulse functions to yield the generalized block pulse operational matrix F^ν as

$$(I^\nu \psi_m)(t) = F^\nu \psi_m(t) \tag{13}$$

where

$$F^\nu = (\frac{b}{m})^\nu \frac{1}{\Gamma(\nu+2)} \begin{pmatrix} 1 & \xi_2 & \xi_3 & \dots & \xi_m \\ 0 & 1 & \xi_2 & \dots & \xi_{m-1} \\ 0 & 0 & 1 & \dots & \xi_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\xi_1 = 1$, $\xi_p = p^{\nu+1} - 2(p-1)^{\nu+1} + (p-2)^{\nu+1}$ $(p = 2, 3, 4, ldots, m - i + 1)$ For further details see refs. [39]. The Haar functions are piecewise constant, so it may be expanded into an $m-$term block pulse functions (BPF) as

$$H_m(t) = H_{m\times m}\psi_m(t) \tag{14}$$

In [31] Haar wavelets operational matrix of fractional order integration is derived by

$$(I^\nu H_m)(t) \approx P^\nu H_m(t) \tag{15}$$

where P^ν is $m \times m$ order Haar wavelets operational matrix of fractional order integration. Substituting Eq.(2) in Eq.(14) we get

$$\begin{aligned}(I^\nu H_m)(t) &\approx (I^\nu H_{m\times m}\psi_m)(t) = H_{m\times m}(I^\nu \psi_m)(t) \\ &\approx H_{m\times m}F^\nu \psi_m(t) \end{aligned} \tag{16}$$

From Eq.(15) and Eq.(16), it can be written as:

$$P^\nu H_m(t) = H_{m\times m}F^\nu \tag{17}$$

Therefore, P^ν can be obtained as

$$P^\nu = H.F^\nu.H^{-1} \tag{18}$$

C. Multi Resolution Analysis (MRA)

Any space V can be constructed using a basis function $h(2^m t)$ as:

$$V_m = span\{h(2^m t - n)\}_{n,m\in Z}$$

$h(t)$ is called scaling function, also known as 'Father function'. The chain of subspaces $ldots V_{-2}, V_{-1}, V_0, V_1, V_2 ldots$ with the following axioms is called multi-resolution analysis (MRA) [28].

$$i)\{\overline{\bigcup V_m}\}_{m\in Z} = L_2(R)$$

$$ii)\{\bigcap V_m\}_{m\in Z} = \{0\}$$

$$iii)\text{There exists } h(t) \text{such that,} V_0 = span\{h(t-n)\}_{n\in Z}$$

$$iv)\{h(t-n)\}_{n\in Z}\text{is an orthogonal set.}$$

$$v)\text{If} f(t) \in V_m \text{ then } f(2^{-m}t) \in V_0,\ \forall\ m \in Z)$$

$$vi)\text{If} f(t) \in V_0 \text{ then } f(t-n) \in V_0,\ \forall\ n \in Z) \tag{19}$$

Under the given axioms, there exists a $\psi(.) \in L_2(R)$, such that $\{\psi(2^m t - n)\}_{m,n\in Z}$ spans $L_2(R)$. The wavelet function $\psi(.)$ is also called 'Mother wavelet'.

Convergence of the method

Let $\frac{\partial^3 u(x,t)}{\partial t\partial x^2}$ and $\frac{\partial^3 v(x,t)}{\partial t\partial x^2}$ are continuous and bounded functions on $(0,1)\times(0,1)$, then $\exists M_1, M_2 > 0, \forall x, t \in (0,1)\times(0,1)$

$$|\frac{\partial^3 u(x,t)}{\partial t\partial x^2}| \le M_1 \text{and } |\frac{\partial^3 v(x,t)}{\partial t\partial x^2}| \le M_2 \tag{20}$$

Let $u_m(x,t)$ and $v_m(x,t)$ are the following approximations of $u(x,t)$ and $v(x,t)$,

$$\begin{aligned} u_m(x,t) &\approx \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} u_{ij}h_i(x)h_j(t) \text{ and} \\ v_m(x,t) &\approx \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} v_{ij}h_i(x)h_j(t) \end{aligned} \tag{21}$$

then we have

$$\begin{aligned} u(x,t) - u_m(x,t) &= \sum_{i=m}^{\infty}\sum_{j=m}^{\infty} u_{ij}h_i(x)h_j(t) \\ &= \sum_{i=2^{p+1}}^{\infty}\sum_{j=2^{p+1}}^{\infty} u_{ij}h_i(x)h_j(t) \text{ and} \\ v(x,t) - v_m(x,t) &= \sum_{i=m}^{\infty}\sum_{j=m}^{\infty} v_{ij}h_i(x)h_j(t) \\ &= \sum_{i=2^{p+1}}^{\infty}\sum_{j=2^{p+1}}^{\infty} v_{ij}h_i(x)h_j(t) \end{aligned} \tag{22}$$

Theorem 1: Let the functions $u_m(x,t)$ and $v_m(x,t)$ obtained by using Haar wavelets are the approximation of $u(x,t)$ and $v(x,t)$ then we have the errors bounded as following

$$\|u(x,t)-u_m(x,t)\|_E \le \frac{M_1}{\sqrt{3}m^3} \text{ and }$$
$$\|v(x,t)-v_m(x,t)\|_E \le \frac{M_2}{\sqrt{3}m^3} \tag{23}$$

where

$$\|u(x,t)\|_E = (\int_0^1\int_0^1 u^2(x,t)dxdt)^{1/2} \text{ and }$$
$$\|v(x,t)\|_E = (\int_0^1\int_0^1 v^2(x,t)dxdt)^{1/2} \tag{24}$$

See proof in [32].

III. The Schrödinger equations

A. The time-fractional Schrödinger equation

The time-fractional Schrödinger equation (T-FSE) has the following form

$$i\frac{\partial^\nu \phi(x,t)}{\partial t^\nu} + \lambda\frac{\partial^2\phi(x,t)}{\partial x^2} + \eta \mid \phi \mid^2 \phi + \mu(x)\phi = q(x,t),$$
$$0 < x,t \le 1 \tag{25}$$

with initial conditions $\phi(x,0) = f(x), \phi(0,t) = g(t)$, $\phi^{'}(0,t) = h(t)$
where $0 < \nu \le 1, \lambda$ and η are real constants, $\mu(x)$ is the trapping potential and $\phi(x,t), f(x), g(t), h(t)$ and $q(x,t)$ are complex functions. We can express complex functions $\phi(x,t), f(x), g(t), h(t)$ and $q(x,t)$ into their respective real and imaginary parts as

$$\begin{aligned} \phi(x,t) &= u(x,t)+iv(x,t)\\ \mu(x) &= \mu_1(x)+i\mu_2\\ f(x) &= f_1(x)+if_2(x)\\ g(t) &= g_1(t)+ig_2(t)\\ h(t) &= h_1(t)+ih_2(t)\\ q(x,t) &= q_1(x,t)+iq_2(x,t) \end{aligned} \tag{26}$$

Substituting Eq. (26) into Eq. (25) and collecting real and imaginary parts, then Eq. (25) can be written as coupled time-fractional nonlinear partial differential equations as:

$$\begin{aligned} &-\frac{\partial^\nu v(x,t)}{\partial t^\nu} + \lambda\frac{\partial^2 u(x,t)}{\partial x^2} + \eta(u^2+v^2)u + \mu_1(x)\\ &\qquad u(x,t) - q_1(x,t) = 0\\ &\frac{\partial^\nu u(x,t)}{\partial t^\nu} + \lambda\frac{\partial^2 v(x,t)}{\partial x^2} + \eta(u^2+v^2)v + \mu_2(x)\\ &\qquad v(x,t) - q_2(x,t) = 0 \end{aligned} \tag{27}$$

with initial conditions

$$\begin{aligned} &u(x,0) = f_1(x), v(x,0) = f_2(x), u(0,t) = g_1(t),\\ &v(0,t) = g_2(t), u^{'}(0,t) = h_1(t), v^{'}(0,t) = h_2(t) \end{aligned} \tag{28}$$

B. The time-fractional coupled Schrödinger system

The time-fractional coupled Schrödinger system (T-FCSS) has the following form

$$\begin{aligned} &i\frac{\partial^\nu \phi(x,t)}{\partial t^\nu} + i\frac{\partial^2\phi(x,t)}{\partial x^2} + \frac{\partial^2\phi(x,t)}{\partial x^2} + \lambda_1(\mid \phi \mid^2 + \mid \psi \mid^2)\\ &\phi(x,t) + \alpha_1(x)\phi(x,t) + \beta_1\psi(x,t) - f_1(x,t) = 0\\ &i\frac{\partial^\nu \psi(x,t)}{\partial t^\nu} + i\frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{\partial^2\psi(x,t)}{\partial x^2} + \lambda_2(\mid \phi \mid^2 + \mid \psi \mid^2)\\ &\psi(x,t) + \alpha_2(x)\phi(x,t) + \beta_2\psi(x,t) - f_2(x,t) = 0 \end{aligned} \tag{29}$$

with initial conditions

$$\begin{aligned} &\phi(x,0) = f_3(x), \phi(0,t) = g_3(t), \phi^{'}(0,t) = h_3(t),\\ &\psi(x,0) = f_4(x), \psi(0,t) = g_4(t), \psi^{'}(0,t) = h_4(t) \end{aligned} \tag{30}$$

where $0 < \nu \le 1, \lambda_1, \lambda_2, \alpha_1, \alpha_2, \beta_1$ andβ_2 are real constants and $\phi(x,t), \psi(x,t), f_3(x), f_4(t), g_3(t), g_4(t), h_3(t), h_4(t)$, $q_1(x,t)$ and $q_2(x,t)$ are complex functions. We can express complex functions into their respective real and imaginary parts as

$$\begin{aligned} &\phi(x,t) = u(x,t)+iv(x,t),\ \psi(x,t) = r(x,t)+is(x,t)\\ &f_3(x) = f_5(x)+if_6(x),\ f_4(x) = f_7(x)+if_8(x)\\ &g_3(t) = g_5(t)+ig_6(t),\ g_4(t) = g_7(t)+ig_8(t)\\ &h_3(t) = h_5(t)+ih_6(t),\ h_4(t) = h_7(t)+ih_8(t)\\ &q_3(x,t) = q_5(x,t)+iq_6(x,t),\ q_4(x,t) = q_7(x,t)+iq_8(x,t) \end{aligned} \tag{31}$$

Substituting Eq. (31) into Eq. (29) and equating real and imaginary parts we get the system of two coupled time-fractional nonlinear partial differential equations.

$$\begin{aligned} &-\frac{\partial^\nu v(x,t)}{\partial t^\nu} - \frac{\partial v(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2} + \lambda_1(u^2+v^2+r^2+s^2)\\ &u(x,t) + \alpha_1 u(x,t) + \beta_1 r(x,t) - q_3(x,t) = 0\\ &\frac{\partial^\nu u(x,t)}{\partial t^\nu} + \frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 v(x,t)}{\partial x^2} + \lambda_1(u^2+v^2+r^2+s^2)\\ &v(x,t) + \alpha_1 v(x,t) + \beta_1 s(x,t) - q_4(x,t) = 0\\ &-\frac{\partial^\nu s(x,t)}{\partial t^\nu} + \frac{\partial s(x,t)}{\partial x} + \frac{\partial^2 r(x,t)}{\partial x^2} + \lambda_2(u^2+v^2+r^2+s^2)\\ &r(x,t) + \alpha_2 u(x,t) + \beta_2 r(x,t) - q_5(x,t) = 0\\ &\frac{\partial^\nu r(x,t)}{\partial t^\nu} - \frac{\partial r(x,t)}{\partial x} + \frac{\partial^2 s(x,t)}{\partial x^2} + \lambda_2(u^2+v^2+r^2+s^2)\\ &s(x,t) + \alpha_2 v(x,t) + \beta_2 s(x,t) - q_6(x,t) = 0 \end{aligned} \tag{32}$$

IV. The proposed method

A. For time-fractional Schrödinger equation

Any arbitrary function $u(x,t) \in L_2([0,1)\times[0,1))$ and $v(x,t) \in L_2([0,1)\times[0,1))$, can be expanded into Haar series [32] as:

$$\dot{u}^{''}(x,t) = \sum_{i=1}^{2m}\sum_{j=1}^{2m} u_{ij}h_i(x)h_j(t)$$
$$\dot{v}^{''}(x,t) = \sum_{i=1}^{2m}\sum_{j=1}^{2m} v_{ij}h_i(x)h_j(t) \tag{33}$$

where $2M \times 2M$ Haar coefficient matrix of u_{ij} and v_{ij} in Eq. (33) can be written as:

$$\begin{aligned} \dot{\boldsymbol{U}}^{''} &= H^T(x).U.H(t) \\ \dot{\boldsymbol{V}}^{''} &= H^T(x).V.H(t) \end{aligned} \tag{34}$$

Let dots and primes in Eq. (33) represent differentiation with respect to t and x, respectively. By integrating Eq. (34) with respect to t from 0 to t, we get

$$\begin{aligned} \boldsymbol{U}^{''} &= H^T(x).U.P^1.H(t) + u^{''}(x,0) \\ \boldsymbol{V}^{''} &= H^T(x).V.P^1.H(t) + v^{''}(x,0) \end{aligned} \tag{35}$$

on integrating Eq. (35) twice with respect to x from 0 to x, then we get

$$\begin{aligned} \boldsymbol{U}^{'} &= H^T(x).[P^1]^T.U.P^1.H(t) + u^{'}(x,0) - u^{'}(0,0) + \\ &\qquad u^{'}(0,t) \\ \boldsymbol{V}^{'} &= H^T(x).[P^1]^T.V.P^1.H(t) + v^{'}(x,0) - v^{'}(0,0) + \\ &\qquad v^{'}(0,t) \end{aligned} \tag{36}$$

and then

$$\begin{aligned} \boldsymbol{U} &= H^T(x).[P^2]^T.U.P^1.H(t) + u(x,0) - u(0,0) - \\ &\qquad xu^{'}(0,0) + xu^{'}(0,t) + u(0,t) \\ \boldsymbol{V} &= H^T(x).[P^2]^T.V.P^1.H(t) + v(x,0) - v(0,0) - \\ &\qquad xv^{'}(0,0) + xv^{'}(0,t) + v(0,t) \end{aligned} \tag{37}$$

Applying differential operator D_t^{ν} on both sides of Eq. (37) and using property (ii) of Eq. (4)

$$\begin{aligned} D_t^{\nu}\boldsymbol{U} &= H^T(x).[P^2]^T.U.P^{1-\nu}.H(t) + xD_t^{\nu}u^{'}(0,t) + \\ &\qquad D_t^{\nu}u(0,t) \\ D_t^{\nu}\boldsymbol{V} &= H^T(x).[P^2]^T.V.P^{1-\nu}.H(t) + xD_t^{\nu}v^{'}(0,t) + \\ &\qquad D_t^{\nu}v(0,t) \end{aligned} \tag{38}$$

Substitution of Eqs. (35), (37) and (38) into Eq. (27) may lead to coupled system of time-fractional differential equations. This system will have some unknown functions $u^{''}(x,0), u^{'}(x,0), u^{'}(0,0), u(0,0), D_t^{\nu}u^{'}(0,t), D_t^{\nu}u(0,t),$ $v^{''}(x,0), v^{'}(x,0), v^{'}(0,0), v(0,0), D_t^{\nu}v^{'}(0,t),$ and $D_t^{\nu}v(0,t)$. With the help of initial conditions all these functions are calculated. We solve Eq. (27) for unknown Haar coefficients by using collocation method. Finally, for Haar solution of T-FSE, we substitute values of Haar coefficients in Eq. (37).

B. For time-fractional coupled Schrödinger system

Now, we approximate $u(x,t), v(x,t), r(x,t)$ and $s(x,t)$, by the Haar series [32] as:

$$\begin{aligned} \dot{u}^{''}(x,t) &= \sum_{i=1}^{2m}\sum_{j=1}^{2m} u_{ij}h_i(x)h_j(t) \\ \dot{v}^{''}(x,t) &= \sum_{i=1}^{2m}\sum_{j=1}^{2m} v_{ij}h_i(x)h_j(t) \\ \dot{r}^{''}(x,t) &= \sum_{i=1}^{2m}\sum_{j=1}^{2m} r_{ij}h_i(x)h_j(t) \\ \dot{s}^{''}(x,t) &= \sum_{i=1}^{2m}\sum_{j=1}^{2m} s_{ij}h_i(x)h_j(t) \end{aligned} \tag{39}$$

Matrix form of Eq. (39) can be written as:

$$\begin{aligned} \dot{\boldsymbol{U}}^{''} &= H^T(x).U.H(t) \\ \dot{\boldsymbol{V}}^{''} &= H^T(x).V.H(t) \\ \dot{\boldsymbol{R}}^{''} &= H^T(x).R.H(t) \\ \dot{\boldsymbol{S}}^{''} &= H^T(x).S.H(t) \end{aligned} \tag{40}$$

Let dots and primes in Eq. (40) represent differentiation with respect to t and x, respectively. By integrating Eq. (40) with respect to t from 0 to t, we get

$$\begin{aligned} \boldsymbol{U}^{''} &= H^T(x).U.P^1.H(t) + u^{''}(x,0) \\ \boldsymbol{V}^{''} &= H^T(x).V.P^1.H(t) + v^{''}(x,0) \\ \boldsymbol{R}^{''} &= H^T(x).R.P^1.H(t) + r^{''}(x,0) \\ \boldsymbol{S}^{''} &= H^T(x).S.P^1.H(t) + s^{''}(x,0) \end{aligned} \tag{41}$$

on integrating Eq. (41) twice with respect to x from 0 to x, once we get

$$\begin{aligned} \boldsymbol{U}^{'} &= H^T(x).[P^1]^T.U.P^1.H(t) + u^{'}(x,0) - u^{'}(0,0) + \\ &\qquad u^{'}(0,t) \\ \boldsymbol{V}^{'} &= H^T(x).[P^1]^T.V.P^1.H(t) + v^{'}(x,0) - v^{'}(0,0) + \\ &\qquad v^{'}(0,t) \\ \boldsymbol{R}^{'} &= H^T(x).[P^1]^T.R.P^1.H(t) + r^{'}(x,0) - r^{'}(0,0) + \\ &\qquad r^{'}(0,t) \\ \boldsymbol{S}^{'} &= H^T(x).[P^1]^T.S.P^1.H(t) + s^{'}(x,0) - s^{'}(0,0) + \\ &\qquad s^{'}(0,t) \end{aligned} \tag{42}$$

and

$$\begin{aligned} \boldsymbol{U} &= H^T(x).[P^2]^T.U.P^1.H(t) + u(x,0) - u(0,0) - \\ &\qquad xu^{'}(0,0) + xu^{'}(0,t) + u(0,t) \\ \boldsymbol{V} &= H^T(x).[P^2]^T.V.P^1.H(t) + v(x,0) - v(0,0) - \\ &\qquad xv^{'}(0,0) + xv^{'}(0,t) + v(0,t) \\ \boldsymbol{R} &= H^T(x).[P^2]^T.R.P^1.H(t) + r(x,0) - r(0,0) - \\ &\qquad xr^{'}(0,0) + xr^{'}(0,t) + r(0,t) \\ \boldsymbol{S} &= H^T(x).[P^2]^T.S.P^1.H(t) + s(x,0) - s(0,0) - \\ &\qquad xs^{'}(0,0) + xs^{'}(0,t) + s(0,t) \end{aligned} \tag{43}$$

Applying differential operator D_t^{ν} on both sides of Eq. (43) and using property (ii) of Eq. (4)

$$\begin{aligned} D_t^{\nu}\boldsymbol{U} &= H^T(x).[P^2]^T.U.P^{1-\nu}.H(t) + xD_t^{\nu}u^{'}(0,t) + \\ &\qquad D_t^{\nu}u(0,t) \\ D_t^{\nu}\boldsymbol{V} &= H^T(x).[P^2]^T.V.P^{1-\nu}.H(t) + xD_t^{\nu}v^{'}(0,t) + \\ &\qquad D_t^{\nu}v(0,t) \\ D_t^{\nu}\boldsymbol{R} &= H^T(x).[P^2]^T.R.P^{1-\nu}.H(t) + xD_t^{\nu}r^{'}(0,t) + \\ &\qquad D_t^{\nu}r(0,t) \\ D_t^{\nu}\boldsymbol{S} &= H^T(x).[P^2]^T.S.P^{1-\nu}.H(t) + xD_t^{\nu}s^{'}(0,t) + \\ &\qquad D_t^{\nu}s(0,t) \end{aligned} \tag{44}$$

Substitution of Eqs. (41), (42), (43) and (44) into Eq. (32) may lead to the two coupled systems of time-fractional differential equations. This system has some unknown functions. With the help of initial conditions all these functions are calculated. We solve system of Eq. (32) for unknown Haar coefficients by using collocation method. Finally, for Haar solution of time-fractional coupled Schrödinger system (T-FCSS), we substitute values of Haar coefficients in Eq. (43).

V. Numerical Problems

In this section, four test problems are taken to test the efficiency and accuracy of the proposed scheme. The computations associated with the problems were executed using *Mathematica 10*.

Problem 1: Consider the linear T-FSE which is also found in [6] with

$$\lambda = 1, \eta = 0, \mu(x) = 0 \text{ and } q(x,t) = \left(\frac{2it^{2-\nu}}{\Gamma(3-\nu)} - t^2\right)e^{ix} \quad (45)$$

Subjected to initial conditions

$$\phi(x,0) = 0, \phi(0,t) = t^2, \phi^{'}(0,t) = it^2 \quad (46)$$

the exact solution for $\nu = 1$ is

$$\phi(x,t) = t^2 e^{ix} \quad (47)$$

Substitute Eq. (45) in Eq. (25) the determined coupled system of equations is

$$\begin{aligned} -\frac{\partial^\nu v(x,t)}{\partial t^\nu} + \frac{\partial^2 u(x,t)}{\partial x^2} - (t^2\cos x + \frac{2t^{2-\nu}\sin x}{\Gamma(3-\nu)}) = 0 \\ \frac{\partial^\nu u(x,t)}{\partial t^\nu} + \frac{\partial^2 v(x,t)}{\partial x^2} - (t^2\sin x - \frac{2t^{2-\nu}\cos x}{\Gamma(3-\nu)}) = 0 \quad (48) \end{aligned}$$

By using the method given in Section IV A, Eq. (48) can be written as

$$\begin{aligned} -H^T(x).[P^2]^T.V.P^{1-\nu}.H(t) + H^T(x).U.P^1.H(t) + \\ \frac{\Gamma(\nu+1)xt^{2-\nu}}{\Gamma(2-\nu+1)} - (t^2\cos x + \frac{2t^{2-\nu}\sin x}{\Gamma(3-\nu)}) = 0 \\ H^T(x).[P^2]^T.U.P^{1-\nu}.H(t) + H^T(x).V.P^1.H(t) + \\ \frac{\Gamma(\nu+1)t^{2-\nu}}{\Gamma(2-\nu+1)} - t^2\sin x + \frac{2t^{2-\nu}\cos x}{\Gamma(3-\nu)}) = 0 \quad (49) \end{aligned}$$

Towards the approximate solution, we first collocate Eq. (49) at points

$$x_i = \frac{i-0.5}{2m}, \; t_j = \frac{j-0.5}{2m} \quad (50)$$

Eq. (49) $\Rightarrow$

$$\begin{aligned} -H^T(x_i).[P^2]^T.V.P^{1-\nu}.H(t_j) + H^T(x_i).U.P^1.H(t_j) + \\ \frac{\Gamma(\nu+1)x_i t_j^{2-\nu}}{\Gamma(2-\nu+1)} - (t_j^2\cos x_i + \frac{2t_j^{2-\nu}\sin x_i}{\Gamma(3-\nu)}) = 0 \\ H^T(x_i).[P^2]^T.U.P^{1-\nu}.H(t_j) + H^T(x_i).V.P^1.H(t_j) + \\ \frac{\Gamma(\nu+1)t_j^{2-\nu}}{\Gamma(2-\nu+1)} - t_j^2\sin x_i + \frac{2t_j^{2-\nu}\cos x_i}{\Gamma(3-\nu)}) = 0 \quad (51) \end{aligned}$$

Eq. (51) generates two systems of $2M$ algebraic equations of Haar coefficients. The values of Haar coefficients are obtained from system of Eq. (51) by using Newton's iterative method. With the help of these coefficients, Haar solutions are attained from Eq. (37). Comparison of the Haar solutions by Homotopy analysis method in [5] is shown in Table I with different values of ν.

TABLE I
Comparison Between Haar Solutions ($J = 1, m = 4$) and HAM [5] of Problem 1

t	x	$\nu = 0.1$		$\nu = 0.3$		$\nu = 0.5$	
		HWCM	**HAM**[5]	**HWCM**	**HAM**[5]	**HWCM**	**HAM**[5]
0.125	0.125	0.015584	0.015729	0.015588	0.018166	0.015602	0.032192
	0.375	0.140258	0.140320	0.140277	0.126725	0.140322	0.18097
	0.625	0.389603	0.391911	0.389639	0.331403	0.389698	0.417840
	0.875	0.763621	0.773331	0.763678	0.642092	0.763759	0.735157
0.375	0.125	0.015639	0.014928	0.015909	0.015876	0.016599	0.029901
	0.375	0.140697	0.137242	0.142257	0.117591	0.144873	0.171424
	0.625	0.390743	0.387923	0.394153	0.321101	0.398660	0.407056
	0.875	0.765763	0.770778	0.771532	0.639027	0.778130	0.736778
0.625	0.125	0.015881	0.014367	0.017251	0.013413	0.020637	0.026391
	0.375	0.142657	0.135328	0.150692	0.110280	0.164111	0.159805
	0.625	0.395831	0.385461	0.413348	0.315020	0.436643	0.396412
	0.875	0.775363	0.768819	0.805018	0.639973	0.839202	0.740576
0.875	0.125	0.016431	0.014212	0.020145	0.011454	0.028487	0.022180
	0.375	0.147236	0.135101	0.169643	0.107096	0.205320	0.148820
	0.625	0.407697	0.385159	0.456638	0.314912	0.519962	0.388621
	0.875	0.797810	0.767939	0.880970	0.644682	0.974802	0.745598

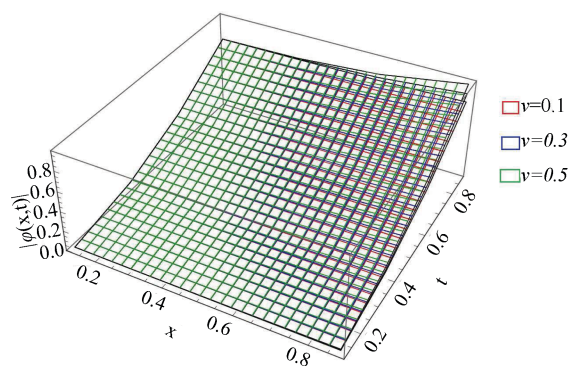

Fig. 1. Haar solutions of Problem 1 at $\nu = 0.1, 0.3, 0.5$

Problem 2: Consider a nonlinear cubic form of T-FSE [6] with

$$\lambda = 1, \eta = 1, \mu(x) = 0 \text{ and}$$
$$q(x,t) = \left(-\frac{2t^{2-\nu}}{\Gamma(3-\nu)} + (-4\pi^2t^2 + t^6)i\right)e^{-2\pi ix} \quad (52)$$

and initial conditions

$$\phi(x,0) = 0, \phi(0,t) = it^2, \phi^{'}(0,t) = 2\pi t^2 \quad (53)$$

with exact solution for $\nu = 1$ is

$$\phi(x,t) = t^2 ie^{-2\pi ix} \quad (54)$$

Substitute Eq. (52) in Eq. (25) the determine coupled system of equations is

$$-\frac{\partial^\nu v(x,t)}{\partial t^\nu}+\frac{\partial^2 u(x,t)}{\partial x^2}+(u^2+v^2)u-((t^6-4\pi^2t^2)\sin 2\pi x-\frac{2t^{3-\nu}}{\Gamma(3-\nu)}\cos 2\pi x)=0 \quad (55)$$

$$\frac{\partial^\nu u(x,t)}{\partial t^\nu}+\frac{\partial^2 v(x,t)}{\partial x^2}+(u^2+v^2)v-((t^6-4\pi^2t^2)\cos 2\pi x+\frac{2t^{3-\nu}}{\Gamma(3-\nu)}\sin 2\pi x)=0 \quad (56)$$

On following the method described in Section IV A, we have

$$-H^T(x).[P^2]^T.V.P^{1-\nu}.H(t)+H^T(x).U.P^1.H(t)+((H^T(x).[P^2]^T.U.P^1.H(t)+2\pi t^2x)^2+(H^T(x).[P^2]^T.V.P^1.H(t)+t^2)^2).(H^T(x).[P^2]^T.U.P^1.H(t)+2\pi t^2x)+\frac{\Gamma(3)}{\Gamma(3-\nu)}t^{2-\nu}-((t^6-4\pi^2t^2)\sin 2\pi x-\frac{2t^{3-\nu}}{\Gamma(3-\nu)}\cos 2\pi x)=0,$$
$$H^T(x).[P^2]^T.U.P^{1-\nu}.H(t)+H^T(x).V.P^1.H(t)+((H^T(x).[P^2]^T.U.P^1.H(t)+2\pi t^2x)^2+(H^T(x).[P^2]^T.V.P^1.H(t)+t^2)^2).(H^T(x).[P^2]^T.V.P^1.H(t)+t^2)+\frac{2\pi\Gamma(3)}{\Gamma(3-\nu)}xt^{2-\nu}-((t^6-4\pi^2t^2)\cos 2\pi x+\frac{2t^{3-\nu}}{\Gamma(3-\nu)}\sin 2\pi x)=0 \quad (57)$$

For approximate solution of Eq. (56), putting collocation points of Eq. (50) in above equations generates two systems of $2M$ non linear algebraic equations of Haar coefficients. The values of Haar coefficients are obtained by using Newton's iterative method and then with the help of these coefficients Haar solutions are attained from Eq. (37). The Haar solutions comparisone with the method in [5] for the different values of ν is shown in Table II.

TABLE II
COMPARISON BETWEEN HAAR SOLUTIONS ($J=1, m=4$) AND HAM [5] OF PROBLEM 2

t	x	$\nu=0.1$ HWCM	$\nu=0.1$ HAM[5]	$\nu=0.5$ HWCM	$\nu=0.5$ HAM[5]	$\nu=0.9$ HWCM	$\nu=0.9$ HAM[5]
0.125	0.125	0.013539	0.013276	0.013578	0.006316	0.013768	0.002593
	0.375	0.121912	0.132538	0.121985	0.097840	0.122023	0.062856
	0.625	0.339272	0.385008	0.339565	0.347687	0.340151	0.276693
	0.875	0.668433	0.770812	0.668861	0.793325	0.669170	0.730876
0.375	0.125	0.010136	0.013277	0.012634	0.006316	0.020531	0.002593
	0.375	0.087449	0.132538	0.091611	0.097839	0.097166	0.062855
	0.625	0.235941	0.385008	0.240394	0.347687	0.248868	0.276693
	0.875	0.450485	0.770812	0.449529	0.793327	0.445334	0.730872
0.625	0.125	0.016209	0.013277	0.018019	0.006316	0.037906	0.002600
	0.375	0.144228	0.132538	0.142259	0.097840	0.133228	0.062856
	0.625	0.391331	0.385008	0.383766	0.347687	0.379218	0.276693
	0.875	0.723229	0.770812	0.709533	0.793325	0.681230	0.730876
0.875	0.125	0.016724	0.013276	0.011634	0.006316	0.064282	0.002600
	0.375	0.161766	0.132538	0.166036	0.097840	0.180905	0.062856
	0.625	0.448134	0.385008	0.430220	0.347687	0.413375	0.276693
	0.875	0.856867	0.770812	0.860240	0.793327	0.847833	0.730872

Problem 3: Consider T-FSE with trapping potential [6] for

$\lambda=1, \eta=1, \mu(x)=\cos^2 x$ and

$$q(x,t)=(i\frac{6t^{3-\nu}}{\Gamma(4-\nu)}-\frac{1}{4}t^3+t^9+t^3\cos^2 x)e^{\frac{ix}{2}} \quad (58)$$

Subjected to initial conditions

$$\phi(x,0)=0, \phi(0,t)=t, \phi^{'}(0,t)=i\frac{t^3}{2} \quad (59)$$

with exact solution for $\nu=1$ is

$$\phi(x,t)=t^3e^{\frac{ix}{2}} \quad (60)$$

Determine the coupled system of equations by substituting Eq. (57) in Eq. (25). Next, following the method illustrated in Section IV A, by following same steps of *problem 1* Haar solutions are attained. The obtained Haar solutions are compared with the Homotopy analysis method [5] and at different values of ν are shown in Table III.

TABLE III
COMPARISON BETWEEN HAAR SOLUTIONS ($J=1, m=4$) AND HAM [5] OF PROBLEM 3

t	x	$\nu=0.1$ HWCM	$\nu=0.1$ HAM[5]	$\nu=0.3$ HWCM	$\nu=0.3$ HAM[5]	$\nu=0.5$ HWCM	$\nu=0.5$ HAM [5]
0.125	0.125	0.002666	0.001164	0.005127	0.001923	0.009737	0.003186
	0.375	0.064478	0.032624	0.099565	0.043939	0.151781	0.100635
	0.625	0.283647	0.155864	0.395466	0.219652	0.544319	0.438394
	0.875	0.752604	0.482083	0.981003	0.706422	1.262370	1.267630
0.375	0.125	0.002335	0.001064	0.004453	0.001859	0.008360	0.002993
	0.375	0.056487	0.029282	0.086504	0.040986	0.130392	0.082892
	0.625	0.248554	0.138363	0.343701	0.159805	0.467832	0.388469
	0.875	0.660306	0.425678	0.853338	0.297603	1.085680	1.080400
0.625	0.125	0.001737	0.000940	0.003234	0.001782	0.005880	0.002751
	0.375	0.042093	0.025192	0.062894	0.046640	0.091797	0.073477
	0.625	0.185563	0.117006	0.250376	0.240900	0.330180	0.329875
	0.875	0.498019	0.355007	0.625716	0.854561	0.769576	0.869481
0.875	0.125	0.000973	0.000833	0.001714	0.001823	0.002981	0.002530
	0.375	0.023635	0.021729	0.033100	0.051878	0.045350	0.065040
	0.625	0.105052	0.098823	0.131553	0.315074	0.158672	0.279519
	0.875	0.299206	0.291128	0.337855	0.257690	0.368943	0.300179

Problem 4: Take into consideration non-linear T-FCSS [35] for

$$\lambda_1 = 2, \lambda_2 = 4, \alpha_1 = \alpha_2 = 1, \beta_1 = 1 \text{ and } \beta_2 = -1$$
$$q_1(x,t) = -\frac{2t^{2-\nu}}{\Gamma(3-\nu)}\sin x + 4t^6\cos x + i(\frac{2t^{2-\nu}}{\Gamma(3-\nu)}\cos x + 4t^6 \sin x) \text{ and} q_2(x,t) = -\frac{2t^{2-\nu}}{\Gamma(3-\nu)}\sin x + 8t^6\cos x + i(\frac{2t^{2-\nu}}{\Gamma(3-\nu)}\cos x + 8t^6\sin x) \quad (61)$$

with initial conditions

$$\phi(x,0) = \psi(x,0) = 0,$$
$$\phi(0,t) = \psi(0,t) = t,$$
$$\phi^{'}(0,t) = \psi^{'}(0,t) = it^2 \quad (62)$$

and with exact solution for these values of λ, α, β and $\nu = 1$ is

$$\phi(x,t) = \psi(x,t) = t^2 e^{ix}, \quad (63)$$

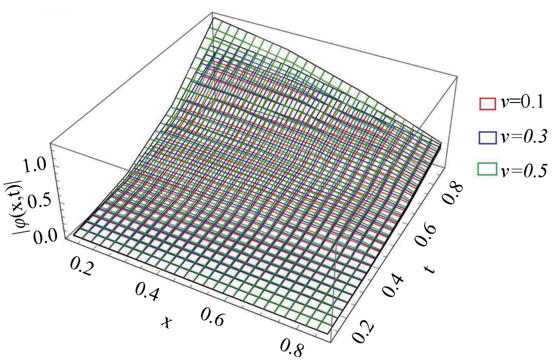

Fig. 2. Haar solutions of Problem 3 at $\nu = 0.1, 0.3, 0.5$

Determine the coupled system of two equations by substituting Eq. (60) in Eq. (29) as

$$-\frac{\partial^\nu v(x,t)}{\partial t^\nu} - \frac{\partial v(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2} + 2(u^2+v^2+r^2+s^2) u(x,t) + u(x,t) + r(x,t) + \frac{2t^{2-\nu}}{\Gamma(3-\nu)}\sin x + 4t^6\cos x = 0,$$
$$\frac{\partial^\nu u(x,t)}{\partial t^\nu} + \frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 v(x,t)}{\partial x^2} + 2(u^2+v^2+r^2+s^2) v(x,t) + v(x,t) + s(x,t) - (\frac{2t^{2-\nu}}{\Gamma(3-\nu)}\cos x + 4t^6\sin x) = 0,$$
$$-\frac{\partial^\nu s(x,t)}{\partial t^\nu} + \frac{\partial s(x,t)}{\partial x} + \frac{\partial^2 r(x,t)}{\partial x^2} + 2(u^2+v^2+r^2+s^2) r(x,t) + u(x,t) - r(x,t) + \frac{2t^{2-\nu}}{\Gamma(3-\nu)}\sin x + 8t^6\cos x = 0,$$
$$\frac{\partial^\nu r(x,t)}{\partial t^\nu} - \frac{\partial r(x,t)}{\partial x} + \frac{\partial^2 s(x,t)}{\partial x^2} + 2(u^2+v^2+r^2+s^2) s(x,t) + v(x,t) - s(x,t) - (\frac{2t^{2-\nu}}{\Gamma(3-\nu)}\cos x + 8t^6\sin x) = 0 \quad (64)$$

Next, following the method illustrated in Section IV B, we have

$$-(H^T(x).[P^2]^T.V.P^{1-\nu}.H(t) + \frac{\Gamma(\nu+1)}{\Gamma(2-\nu+1)}xt^{2-\nu}) - H^T(x).[P^1]^T.V.P^1.H(t) + H^T(x).U.P^1.H(t) + \boldsymbol{U} + 2(\boldsymbol{U}^2+\boldsymbol{V}^2+\boldsymbol{R}^2+\boldsymbol{S}^2)\boldsymbol{U} + \boldsymbol{R} + \frac{2t^{2-\nu}}{\Gamma(3-\nu)}\sin x + 4t^6\cos x = 0, \quad (65)$$

$$H^T(x).[P^2]^T.U.P^{1-\nu}.H(t) + \frac{\Gamma(\nu+1)}{\Gamma(2-\nu+1)}xt^{2-\nu} + H^T(x).[P^1]^T.U.P^1.H(t) + H^T(x).V.P^1.H(t) + \boldsymbol{V} + 2(\boldsymbol{U}^2+\boldsymbol{V}^2+\boldsymbol{R}^2+\boldsymbol{S}^2)\boldsymbol{V} + \boldsymbol{S} - (\frac{2t^{2-\nu}}{\Gamma(3-\nu)}\cos x + 4t^6\sin x) = 0,$$
$$-(H^T(x).[P^2]^T.S.P^{1-\nu}.H(t) + \frac{\Gamma(\nu+1)}{\Gamma(2-\nu+1)}xt^{2-\nu}) + H^T(x).[P^1]^T.S.P^1.H(t) + H^T(x).R.P^1.H(t) + \boldsymbol{U} + 4(\boldsymbol{U}^2+\boldsymbol{V}^2+\boldsymbol{R}^2+\boldsymbol{S}^2)\boldsymbol{R} - \boldsymbol{R} + \frac{2t^{2-\nu}}{\Gamma(3-\nu)}\sin x + 8t^6\cos x = 0,$$
$$(H^T(x).[P^2]^T.R.P^{1-\nu}.H(t) + \frac{\Gamma(\nu+1)}{\Gamma(2-\nu+1)}xt^{2-\nu}) - H^T(x).[P^1]^T.R.P^1.H(t) + H^T(x).S.P^1.H(t) + \boldsymbol{V} + 4(\boldsymbol{U}^2+\boldsymbol{V}^2+\boldsymbol{R}^2+\boldsymbol{S}^2)\boldsymbol{S} - \boldsymbol{S} - (\frac{2t^{2-\nu}}{\Gamma(3-\nu)}\cos x + 8t^6\sin x) = 0 \quad (66)$$

where $\boldsymbol{U} = H^T(x).[P^2]^T.U.P^1.H(t) + t^2$,
$\boldsymbol{V} = H^T(x).[P^2]^T.V.P^1.H(t) + xt^2$,
$\boldsymbol{R} = H^T(x).[P^2]^T.R.P^1.H(t) + t^2$ and
$\boldsymbol{S} = H^T(x).[P^2]^T.S.P^1.H(t) + xt^2$

TABLE IV
COMPARISON BETWEEN HAAR SOLUTIONS ($J = 1, m = 4, \nu = 0.1$) AND HAM [5] OF PROBLEM 4

t	x	$\mid \phi(x,t) \mid$ HWCM	$\mid \phi(x,t) \mid$ HAM [5]	$\mid \psi(x,t) \mid$ HWCM	$\mid \psi(x,t) \mid$ HAM [5]
0.125	0.125	0.015383	0.014440	0.0153167	0.004286
	0.375	0.138501	0.128015	0.137928	0.038537
	0.625	0.385851	0.365216	0.384604	0.143183
	0.875	0.763499	0.949120	0.763355	0.811326
0.375	0.125	0.013949	0.014439	0.013495	0.004286
	0.375	0.125927	0.128015	0.121958	0.038536
	0.625	0.357238	0.365216	0.348492	0.143183
	0.875	0.747914	0.949120	0.746334	0.811326
0.625	0.125	0.011086	0.014440	0.009704	0.004285
	0.375	0.100695	0.128015	0.088716	0.038536
	0.625	0.297240	0.365216	0.272767	0.143183
	0.875	0.697857	0.949120	0.706649	0.811326
0.875	0.125	0.008487	0.014440	0.004160	0.004286
	0.375	0.077508	0.128015	0.040194	0.038537
	0.625	0.236292	0.365216	0.163674	0.143183
	0.875	0.634398	0.949120	0.664360	0.811326

Using Eq. (50), the values of Haar coefficients are obtained and finally with the help of these coefficients Haar solutions are attained from Eq. (43). The Haar solutions are compared with the method in [5] with $\nu = 0.1$ and results are shown in Table IV, and for $\nu = 0.5$ and $\nu = 0.9$ in Table V and Table VI, respectively.

TABLE V
COMPARISON BETWEEN HAAR SOLUTIONS
$(J = 1, m = 4, \nu = 0.5)$ AND HAM [5] OF PROBLEM 4

t	x	$\mid \phi(x,t) \mid$		$\mid \psi(x,t) \mid$	
		HWCM	**HAM** [5]	**HWCM**	**HAM** [5]
0.125	0.125	0.015386	0.015609	0.015323	0.008203
	0.375	0.138522	0.139708	0.137974	0.074131
	0.625	0.385897	0.376762	0.384668	0.225031
	0.875	0.763618	0.919085	0.763415	0.811099
0.375	0.125	0.013951	0.015608	0.013522	0.008203
	0.375	0.125873	0.139708	0.122165	0.074132
	0.625	0.357309	0.376762	0.348789	0.225031
	0.875	0.748433	0.919085	0.746600	0.811099
0.625	0.125	0.011025	0.015609	0.009723	0.008203
	0.375	0.100259	0.139708	0.088875	0.074132
	0.625	0.296585	0.376762	0.272878	0.225031
	0.875	0.696443	0.919085	0.705993	0.811099
0.875	0.125	0.008579	0.015609	0.004118	0.008202
	0.375	0.079810	0.139708	0.040065	0.074131
	0.625	0.235819	0.376762	0.162549	0.225031
	0.875	0.623663	0.919085	0.660364	0.811099

TABLE VI
COMPARISON BETWEEN HAAR SOLUTIONS
$(J = 1, m = 4, \nu = 0.9)$ AND HAM [5] OF PROBLEM 4

t	x	$\mid \phi(x,t) \mid$		$\mid \psi(x,t) \mid$	
		HWCM	**HAM** [5]	**HWCM**	**HAM** [5]
0.125	0.125	0.015395	0.013101	0.015357	0.013101
	0.375	0.138552	0.117724	0.138256	0.118102
	0.625	0.385995	0.322206	0.384873	0.336649
	0.875	0.763808	0.631637	0.763550	0.834047
0.375	0.125	0.013964	0.013101	0.013688	0.013101
	0.375	0.125649	0.117724	0.123396	0.118102
	0.625	0.357809	0.322206	0.349879	0.336649
	0.875	0.749377	0.631637	0.747267	0.834047
0.625	0.125	0.010878	0.013101	0.009803	0.013101
	0.375	0.099177	0.117725	0.089536	0.118102
	0.625	0.296418	0.322206	0.273663	0.336649
	0.875	0.694211	0.631637	0.701977	0.834047
0.875	0.125	0.008446	0.013101	0.003678	0.013101
	0.375	0.085571	0.117725	0.040710	0.118102
	0.625	0.231591	0.322206	0.157683	0.336649
	0.875	0.604013	0.631637	0.640823	0.834047

VI. CONCLUSION

In this paper, we extended the capability of the Haar wavelet collocation method (HWCM) for the solution of time-fractional coupled system of partial differential equations. The main advantage of HWCM is the ability to achieve a good solution and rapid convergence with small number of collocation points. The presence of maximum zeros in the Haar matrices reduces the number of unknown wavelet coefficients that is to be determined, which as a result diminishes the computation time as well. The scheme is tested on some examples of time fractional Schrödinger equations. The presented procedure may very well be extended to solve two dimensional Schrödinger equation and other similar nonlinear problems of partial differential equations of fractional order. The problem discussed here is just for showing the applicability of the proposed computational technique to handle the complex system of differential equation in fractional-order problems in a straight forward way. Also, the Haar wavelet method proves to be capable to efficiently handle the nonlinearity of partial differential equations of fractional order. The main advantages of the proposed algorithm are, its simple application and no requirement of residual or product operational matrix. Numerical solutions for different order of fractional time derivative by Haar wavelet are shown in Tables and Figures. The increasing values of ν show that the solutions are valuable in understanding their respective exact solutions for $\nu = 1$. Comparisons between our approximate solutions of the problems with their actual solutions and with the approximate solutions achieved by a homotopy analysis method [5] confirm the validity and accuracy of our scheme.

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