Decentralized Adaptive Strategies for Synchronization of Fractional-Order Complex Networks

Quan Xu, Shengxian Zhuang, Yingfeng Zeng, and Jian Xiao

***Abstract*—This paper focuses on synchronization of fractional-order complex dynamical networks with decentralized adaptive coupling. Based on local information among neighboring nodes, two fractional-order decentralized adaptive strategies are designed to tune all or only a small fraction of the coupling gains respectively. By constructing quadratic Lyapunov functions and utilizing fractional inequality techniques, Mittag-Leffler function, and Laplace transform, two sufficient conditions are derived for reaching network synchronization by using the proposed adaptive laws. Finally, two numerical examples are given to verify the theoretical results.**

***Index Terms*—Decentralized adaptive control, synchronization, fractional-order complex networks, quadratic Lyapunov functions.**

I. INTRODUCTION

IT is well known that numerous natural and man-made systems can be modeled as complex dynamical networks. Examples include social networks, food webs, epidemic spreading networks, biological networks, scientific citation networks, Internet networks, World Wide Web, electric power grids, and so on[1−3]. In recent years, extensive efforts have been made to understand and study the topology and dynamics of complex networks. Specifically, as a typical collective behavior of complex networks, synchronization has received increasing attention due to its potential applications in many real scenarios[4−5]. So far, many systematic results on different synchronization patterns, such as complete synchronization, lag synchronization, generalized synchronization, cluster synchronization, etc., have been obtained for many kinds of complex networks, see [6-16] and relevant references therein.

To our best knowledge, the results on synchronization mainly concentrated on integer-order complex networks. Nevertheless, it has been recognized that the real objects are generally fractional and fractional calculus allows us to describe and model a real object more accurately than the classical integer-order methods. Not surprisingly, dynamics and control of fractional-order systems has attracted increasing attention from various fields[17−23]. Particularly, synchronization in fractional-order complex networks[24−26] has currently become an interesting and open problem. From a control perspective, the aim here is to find some appropriate controllers such that the controlled fractional-order network is synchronized.

Among them, adaptive control technique has been widely used to synchronize complex networks. In [27-29], many kinds of adaptive strategies were designed to adjust the gains of feedback controllers. Note that, in diffusively coupled networks, nodes are coupled with states difference $x_i - x_j$. This means that a state feedback controller is added to every node. Thus, a network could be synchronized by designing suitable coupling gains among the network nodes. Mathematically, these coupling gains are described by the non-null elements of the weighted time-varying adjacency matrix $G(t)$. Recently, some decentralized adaptive strategies have been used to tune the coupling gains so as to achieve synchronization in complex networks, see [30-34]. Moreover, decentralized adaptive strategies are introduced only to a small fraction of coupling gains[35]. Compared with the centralized adaptive strategies developed in [36,37], the coupling gains are adapted based on local information exchanged among neighboring nodes. However, the synchronization of fractional-order complex networks with decentralized adaptive coupling has never been investigated elsewhere. Therefore, it is important and interesting to study the synchronization of fractional-order complex networks by using the fractional-order decentralized adaptive strategies.

As is known to all, Lyapunov direct method is a standard tool to derive the synchronization criteria for integer-order complex networks. Despite much effort, the Lyapunov-based results about synchronization of integer-order complex networks cannot be directly extended to the fractional-order cases. The main difficulty lies in calculating the fractional derivative of a composite Lyapunov function. For more details about this, one can refer the existing literatures [38,39], in which there were several issues regarding calculation of the fractional derivative of a composite Lyapunov function.

Quite recently, Aguila-Camacho *et al*[40] and Duarte-Mermoud *et al*[41] introduce two lemmas for estimating the Caputo fractional derivative of a quadratic function. Thus,

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Quan Xu is with the School of Electrical Engineering, Southwest Jiaotong University, Chengdu 610031, China, and also with School of Technology, Xihua University, Chengdu 610039, China (e-mail: quanxnjd@sina.com).

Shengxian Zhuang, Yingfeng Zeng, Jian Xiao are with the School of Electrical Engineering, Southwest Jiaotong University, Chengdu 610031, China (e-mail: zhuangsx@sina.com; zyf9999999@163.com; jxiao@home.swjtu.edu.cn).

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one can analyze the stability for fractional-order uncoupled systems and coupled networks by using quadratic Lyapunov functions like classic Lyapunov direct method. But the conditions of fractional Lyapunov direct method[42,43] are relatively conservative and rigorous. As the extensions of Lyapunov direct method, LaSalle's invariance principle, Barbalat's Lemma and other mathematical techniques can be used to solve the adaptive stability problem of integer-order nonlinear systems. However, these tools cannot be directly used in the fractional order case. Thus, additional tools need to be developed, in order to prove the errors convergence in the fractional order case. In this paper, by utilizing Lyapunov functional method combined with fractional inequality techniques, Mittag-Leffler function, and Laplace transform, we study the decentralized adaptive synchronization in fractional-order networks with diffusive coupling.

The remaining of this paper is organized as follows. In Section II, some necessary preliminaries and the model of fractional-order complex networks are given. The main results of this paper are given in Section III. In Section IV, two numerical examples are provided to validate the theoretical results. Finally, some conclusions are presented in Section V.

II. Model description and preliminaries

A. Fractional Calculus and Properties

Definition 1. The Riemann-Liouville fractional integral with $0<\alpha<1$ is given by

$$\mathrm{I}_t^\alpha f(t)= \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau, \tag{1}$$

where $t \geq t_0,\ f(t)$ is an arbitrary integrable function I_t^α is the fractional integral operator, $\Gamma(\cdot)$ is the gamma function $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}\exp(-t)\mathrm{d}t$, and $\exp(\cdot)$ is exponential function.

In this paper, we consider the Caputo definition for fractional derivative, which is most popular in engineering applications because of its advantages[17].

Definition 2. The Caputo fractional derivative with fractional-order $0<\alpha<1$ can be expressed as

$$\mathrm{D}_t^\alpha f(t)= \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-\tau)^{-\alpha}\dot{f}(\tau)\mathrm{d}\tau, \tag{2}$$

where $t\geq t_0$, D_t^α is the Caputo fractional derivative operator. In the following, unless otherwise stated, we consider $\alpha\in(0,1)$.

Moreover, the Laplace transform of Caputo fractional derivative is

$$\mathcal{L}\left\{\mathrm{D}_t^\alpha f(t)\right\}=s^\alpha F(s)-s^{\alpha-1}f(t_0), \tag{3}$$

where $\alpha\in(0,1)$, s denotes the variable in Laplace domain, $\mathcal{L}\{\cdot\}$ is the Laplace transform operator, $F(s)$ is the Laplace transform of $f(t)$ and $f(t_0)$ is the initial value.

Let us pay attention to the following properties of the fractional derivatives[17], which are most commonly used in applications.

Property 1.

$$\mathrm{D}_t^\alpha(ax(t)+by(t))=a\mathrm{D}_t^\alpha x(t)+b\mathrm{D}_t^\alpha y(t). \tag{4}$$

Property 2.

$$\mathrm{I}_t^\alpha\mathrm{D}_t^\alpha f(t)=\ f(t)-f(t_0),\ \ \forall\, t\geq t_0,\ 0<\alpha<1. \tag{5}$$

Property 3. The Caputo fractional derivative of a constant function is always zero.

Definition 3[42,43]. The Mittag-Leffler function with one parameter and two parameters can be defined as

$$\begin{aligned} E_\alpha(z)&=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)},\\ E_{\alpha,\beta}(z)&=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+\beta)}, \end{aligned} \tag{6}$$

where $z\in\mathbb{C},\alpha>0,\beta>0$. Note that $E_{\alpha,1}(z)=E_\alpha(z)$, $E_{1,1}(z)=\exp^z$.

The Laplace transform of Mittag-Leffler function with two parameters can be written as

$$\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}(-kt^\alpha)\right\}=\frac{s^{\alpha-\beta}}{s^\alpha+k},\quad \Re(s)>\mid k\mid^{\frac{1}{\alpha}}, \tag{7}$$

where $t\geq 0$, $\Re(s)$ is the real part of s, $k\in\mathbb{R}$.

A new property for Caputo derivative can be stated in Lemma 1, which can facilitate estimating the fractional derivative of a common quadratic Lyapunov function.

Lemma 1[41]. Let $x(t)\in\mathbb{R}^n$ be a vector of derivable functions. Then, the following inequality holds

$$\mathrm{D}_t^\alpha(x^{\mathrm{T}}(t)Px(t))\leq 2x^{\mathrm{T}}(t)P\mathrm{D}_t^\alpha x(t), \tag{8}$$

where $\alpha\in(0,1]$, $t\geq t_0$ and $P\in\mathbb{R}^{n\times n}$ is a constant, symmetric and positive definite matrix.

B. Network model

Consider a fractional-order complex dynamical network consisting of N identical nodes, which is described by

$$\begin{aligned} \mathrm{D}_t^\alpha x_i(t)=f(t,x_i(t))+c\sum_{j=1}^{N}G_{ij}(t)Ax_j(t),&\\ i=1,2,\cdots,N,& \end{aligned} \tag{9}$$

where $0<\alpha<1$, $x_i=(x_{i1},x_{i2},...,x_{in})\in\mathbb{R}^n$ is the pseudo-state vector of node i, $f:\mathbb{R}^+\times\mathbb{R}^n\rightarrow\mathbb{R}^n$ is a nonlinear vector field, $c>0$ is the coupling strength, $A=\mathrm{diag}(\rho_1,\rho_2,\cdots,\rho_\mathrm{n})\in\mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ is a positive definite inner coupling matrix, $G(t)=(G_{ij}(t))_{N\times N}$ is the time-varying diffusive coupling matrix representing the topological structure of an undirected network. If there is an edge between node i and j at time t, then $G_{ij}(t)=G_{ji}(t)>0$; otherwise $G_{ij}(t)=G_{ji}(t)=0$ $(i\neq j)$, and the diagonal elements of $G(t)$ are defined by

$$G_{ii}=-\sum_{j=1,j\neq i}^{N}G_{ij},\ i=1,2,\cdots,N.$$

Throughout this paper, only connected networks are considered, and $G_{ij}(t),i,j\in\{1,2,\cdots,N\}$ has the same meaning.

Definition 4. The complex network (9) is said to achieve synchronization in the sense that

$$\lim_{t\to\infty}\left\|x_i(t)-\frac{1}{N}\sum_{j=1}^{N}x_j(t)\right\|_2=0,\ i=1,2,\cdots,N, \tag{10}$$

Let $\bar{x}=\frac{1}{N}\sum_{j=1}^{N}x_j$. Then, we get

$$\begin{aligned}\mathrm{D}_t^{\alpha}\bar{x}(t)&=\frac{1}{N}\sum_{j=1}^{N}\mathrm{D}_t^{\alpha}x_j(t)\\&=\frac{1}{N}\sum_{j=1}^{N}\left[f(t,x_j(t))+c\sum_{k=1}^{N}G_{jk}(t)Ax_k(t)\right]\\&=\frac{1}{N}\sum_{j=1}^{N}f(t,x_j(t))+\frac{c}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}G_{ij}(t)Ax_j(t)\\&=\frac{1}{N}\sum_{j=1}^{N}f(t,x_j(t)).\end{aligned}\tag{11}$$

Note that $\frac{c}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}G_{ij}(t)Ax_j(t)=0$ can be obtained from $G_{ij}=G_{ji}$, $G_{ii}=-\sum_{j=1,j\neq i}^{N}G_{ij}$.

Defining$e_i(t)=x_i(t)-\bar{x}(t)$, then the error dynamical network is described as follows:

$$\begin{aligned}\mathrm{D}_t^{\alpha}e_i(t)=&f(t,x_i(t))-\frac{1}{N}\sum_{j=1}^{N}f(t,x_j(t))\\&+c\sum_{j=1}^{N}G_{ij}(t)Ae_j(t),\ i=1,2,\cdots,N.\end{aligned}\tag{12}$$

Assumption 1. The nonlinear function $f(t,x)$ is said to be Lipschitz if there exists a nonnegative constant ε such that $(x-y)^{\mathrm{T}}(f(t,x)-f(t,y))\leq\varepsilon(x-y)^{\mathrm{T}}(x-y)$.

Lemma 2[31]. Let $G=(G_{ij})_{N\times N}$ is a real symmetric and irreducible matrix with

$$G_{ij}=G_{ji}\geq 0(i\neq j),G_{ii}=-\sum_{j=1,j\neq i}^{N}G_{ij}$$

Then,

(1) The eigenvalues of G satisfy

$$\begin{aligned}&0=\lambda_1(G)>\lambda_2(G)\geq\cdots\geq\lambda_N(G),\\&\lambda_2(G)=\max_{x^{\mathrm{T}}1_N=0,x\neq 0}\frac{x^{\mathrm{T}}Gx}{x^{\mathrm{T}}x}.\end{aligned}$$

(2) For any $\eta=(\eta_1,\eta_2,\cdots,\eta_N)^{\mathrm{T}}\in\mathbb{R}^N$

$$\eta^{\mathrm{T}}G\eta=-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}G_{ij}(\eta_i-\eta_j)^2.$$

III. MAIN RESULTS

In this section, two fractional-order decentralized adaptive laws to tune the coupling gains among network nodes are proposed. By utilizing the proposed adaptive strategies, two sufficient conditions are derived to synchronize the proposed fractional-order complex networks.

A. Fractional-order decentralized adaptive strategy for the synchronization

Theorem 1. Suppose that Assumption 1 holds. Then, the network (9) is synchronized under the following fractional-order decentralized adaptive strategy:

$$\begin{aligned}&\mathrm{D}_t^{\alpha}G_{ij}(t)=\gamma_{ij}(x_i(t)-x_j(t))^{\mathrm{T}}A(x_i(t)-x_j(t)),\\&G_{ij}(0)=G_{ji}(0)>0,\end{aligned}\tag{13}$$

$(i,j)\in E$, where E is the set of undirected edges, $\gamma_{ij}=\gamma_{ji}$ are positive constants.

Proof. Construct the Lyapunov functional candidate for system (12) as

$$V_1(t)=\frac{1}{2}\sum_{i=1}^{N}e_i^{\mathrm{T}}(t)e_i(t)+\sum_{i=1}^{N}\sum_{j\in N_i}\frac{c}{4\gamma_{ij}}(G_{ij}(t)-h_{ij})^2,\tag{14}$$

where $h_{ij}=h_{ji}(i\neq j)$ are nonnegative constants, and $h_{ij}=0$ if and only if $G_{ij}(t)=0$.

Applying Lemma 1, the fractional derivative of V_1 along the trajectories of system (12) gives

$$\begin{aligned}\mathrm{D}_{\mathrm{t}}^{\alpha}V_1\leq&\sum_{i=1}^{N}e_i^{\mathrm{T}}(t)\mathrm{D}^{\alpha}e_i(t)\\&+\sum_{i=1}^{N}\sum_{j\in N_i}\frac{c}{2\gamma_{ij}}(G_{ij}(t)-h_{ij})\mathrm{D}^{\alpha}(G_{ij}(t)-h_{ij})\\=&\sum_{i=1}^{N}e_i^{\mathrm{T}}(t)\left[f(t,x_i(t))-\frac{1}{N}\sum_{j=1}^{N}f(t,x_j(t))\right]\\&+c\sum_{i=1}^{N}\sum_{j=1}^{N}e_i^{\mathrm{T}}(t)G_{ij}(t)Ae_j(t)\\&+\sum_{i=1}^{N}\sum_{j\in N_i}\frac{c}{2\gamma_{ij}}(G_{ij}(t)-h_{ij})\mathrm{D}^{\alpha}G_{ij}(t)\\=&\sum_{i=1}^{N}e_i^{\mathrm{T}}(t)\left[f(t,x_i(t))-\frac{1}{N}\sum_{j=1}^{N}f(t,x_j(t))\right]\\&+c\sum_{i=1}^{N}\sum_{j=1}^{N}e_i^{\mathrm{T}}(t)G_{ij}(t)Ae_j(t)\\&+\sum_{i=1}^{N}\sum_{j\in N_i}\frac{c}{2}(G_{ij}(t)-h_{ij})(x_i-x_j)^{\mathrm{T}}A(x_i-x_j)\\=&\sum_{i=1}^{N}e_i^{\mathrm{T}}(t)\left[f(t,x_i(t))-f(t,\ \bar{x})\right]\\&+\sum_{i=1}^{N}e_i^{\mathrm{T}}(t)\left[f(t,\ \bar{x})-\frac{1}{N}\sum_{j=1}^{N}f(t,x_j(t))\right]\\&+c\sum_{i=1}^{N}\sum_{j=1}^{N}e_i^{\mathrm{T}}(t)G_{ij}(t)Ae_j(t)\\&+\sum_{i=1}^{N}\sum_{j\in N_i}\frac{c}{2}(G_{ij}(t)-h_{ij})(e_i-e_j)^{\mathrm{T}}A(e_i-e_j),\end{aligned}\tag{15}$$

Since $\sum_{i=1}^{N} e_i^{\mathrm{T}}(t) = 0$, we have

$$\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\left[f(t,\ \bar{x}) - \frac{1}{N}\sum_{j=1}^{N} f(t, x_j(t))\right] = 0. \quad (16)$$

According to Assumption 1, we can obtain

$$\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\left[f(t, x_i(t)) - f(t, \bar{x})\right] \le \varepsilon \sum_{i=1}^{N} e_i^{\mathrm{T}}(t) e_i(t). \quad (17)$$

Let $H = (h_{ij})_{N\times N}$, where $h_{ii} = -\sum_{j=1, j\neq i}^{N} h_{ij}$. From Lemma 2, we can easily obtain

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{j\in N_i}(G_{ij}(t) - h_{ij})(e_i(t) - e_j(t))^{\mathrm{T}} A (e_i(t) - e_j(t))\\ &\quad = -2\sum_{i=1}^{N}\sum_{j=1}^{N}(G_{ij}(t) - h_{ij}) e_i^{\mathrm{T}}(t) A e_j(t). \end{aligned} \quad (18)$$

Combining (15), (16), (17) and (18), we have

$$\begin{aligned}\mathrm{D}_t^{\alpha} V_1 \le &\varepsilon \sum_{i=1}^{N} e_i^{\mathrm{T}}(t) e_i(t) + c\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^{\mathrm{T}}(t) G_{ij}(t) A e_j(t)\\ &- c\sum_{i=1}^{N}\sum_{j=1}^{N}(G_{ij}(t) - h_{ij}) e_i^{\mathrm{T}}(t) A e_j(t)\\ \le &\varepsilon \sum_{i=1}^{N} e_i^{\mathrm{T}}(t) e_i(t) + c\sum_{i=1}^{N}\sum_{j=1}^{N} h_{ij} e_i^{\mathrm{T}}(t) A e_j(t)\\ = &e^{\mathrm{T}}(t)\left[\varepsilon (I_N \otimes I_n) + c(H\otimes A)\right] e(t), \end{aligned} \quad (19)$$

where $e(t) = (e_1^{\mathrm{T}}(t), e_2^{\mathrm{T}}(t), \cdots, e_N^{\mathrm{T}}(t))^{\mathrm{T}} \in R^{nN}$.

Let $\Lambda = \mathrm{diag}(\lambda_1(H), \lambda_2(H), \cdots, \lambda_N(H))$ be the diagonal matrix associated with H, that is, there exists a unitary matrix $\Phi = (\phi_1, \phi_2, \cdots, \phi_N)$ such that $\Phi^{\mathrm{T}} H \Phi = \Lambda$. Let $y(t) = (y_1^{\mathrm{T}}, y_2^{\mathrm{T}}, \cdots, y_N^{\mathrm{T}})^{\mathrm{T}} = (\Phi^{\mathrm{T}} \otimes I_n) e(t)$. Since $\phi_1 = \frac{1}{\sqrt{N}}(1, 1, \cdots, 1)^{\mathrm{T}}$, one has $y_1(t) = (\phi_1^{\mathrm{T}} \otimes I_n) e(t) = 0$. Then, it follows from (19) that

$$\begin{aligned}\mathrm{D}_t^{\alpha} V_1 &\le \varepsilon e^{\mathrm{T}}(t)(I_N\otimes I_n) e(t)\\ &\quad + c e^{\mathrm{T}}(t)(\Phi\otimes I_n)(\Lambda \otimes A)(\Phi^{\mathrm{T}}\otimes I_n) e(t)\\ &= e^{\mathrm{T}}(t)\left[\varepsilon (I_N\otimes I_n)\right] e(t) + c y^{\mathrm{T}}(t)(\Lambda\otimes A) y(t). \end{aligned} \quad (20)$$

According to the definition of matrix H, one can easily verify that matrix H satisfies the conditions of Lemma 2. Then, by Lemma 2 and since A is positive, we get

$$y^{\mathrm{T}}(t)(\Lambda\otimes A) y(t) \le \lambda_2(H) y^{\mathrm{T}}(t)(I_N\otimes A) y(t). \quad (21)$$

From (20) and (21), it follows that

$$\begin{aligned}\mathrm{D}_t^{\alpha} V_1 \le &\varepsilon e^{\mathrm{T}}(t)(I_N\otimes I_n) e(t)\\ &+ c\lambda_2(H) y^{\mathrm{T}}(t)(I_N\otimes A) y(t)\\ = &\varepsilon e^{\mathrm{T}}(t)(I_N\otimes I_n) e(t)\\ &+ c\lambda_2(H) e^{\mathrm{T}}(t)(\Phi\otimes I_n)(I_N\otimes A)(\Phi^{\mathrm{T}}\otimes I_n) e(t)\\ = &e^{\mathrm{T}}(t)\left[\varepsilon (I_N\otimes I_n) + c\lambda_2(H)(I_N\otimes A)\right] e(t). \end{aligned} \quad (22)$$

Thus, for a given overall coupling strength c, one can choose h_{ij} sufficiently large such that

$$\varepsilon (I_N\otimes I_n) + c\lambda_2(H)(I_N\otimes A) + 1 < 0. \quad (23)$$

Then, it follows from (22) and (23) that

$$\mathrm{D}_t^{\alpha} V_1(t) \le -e^{\mathrm{T}}(t) e(t). \quad (24)$$

There exists a function $m(t) \ge 0$ such that

$$\mathrm{D}_t^{\alpha} V_1(t) + m(t) = -e^{\mathrm{T}}(t) e(t). \quad (25)$$

Applying Laplace transform operator $\mathcal{L}\{\cdot\}$ to (25), we have

$$\mathrm{s}^{\alpha} V_1(\mathrm{s}) - \mathrm{s}^{\alpha-1} V_1(0) + M(s) = -E(s), \quad (26)$$

where the nonnegative constant $V_1(0)$ is the initial value of $V_1(t)$, $V_1(s)$, $M(s)$, and $E(s)$ are the Laplace transforms of $V_1(t)$, $m(t)$, and $e^{\mathrm{T}}(t)e(t)$ respectively.

Since $V_1(t) \ge \frac{1}{2}\sum_{i=1}^{N} e_i^{\mathrm{T}}(t) e_i(t) = \frac{1}{2} e^{\mathrm{T}}(t) e(t)$, there exists a function $n(t) \ge 0$ such that

$$V_1(t) = \frac{1}{2} e^{\mathrm{T}}(t) e(t) + n(t). \quad (27)$$

Applying Laplace transform operator $\mathcal{L}\{\cdot\}$ to (27), we have

$$V_1(\mathrm{s}) = \frac{1}{2} E(s) + N(s), \quad (28)$$

where $N(s)$ is the Laplace transform of $n(t)$.

Combining (26) and (28), we can easily obtain

$$\begin{aligned}E(s) = &\frac{2\mathrm{s}^{\alpha-1}}{\mathrm{s}^{\alpha}+2} V_1(0) - \frac{2\mathrm{s}^{\alpha}}{\mathrm{s}^{\alpha}+2} N(s)\\ &- \frac{2}{\mathrm{s}^{\alpha}+2} M(s), \end{aligned} \quad (29)$$

Taking the Laplace inverse transform of (29), it gives

$$\begin{aligned}e^{\mathrm{T}}(t) e(t) = &2V_1(0) E_{\alpha}(-2t^{\alpha}) - 2n(t) * t^{-1} E_{\alpha,0}(-2t^{\alpha})\\ &- 2m(t) * t^{\alpha-1} E_{\alpha,\alpha}(-2t^{\alpha}). \end{aligned} \quad (30)$$

where $*$ stands the convolution operator.

Since $t^{-1}, t^{\alpha-1}, E_{\alpha,0}(-2t^{\alpha})$, and $E_{\alpha,\alpha}(-2t^{\alpha})$ are nonnegative functions, it follows from (30) that

$$e^{\mathrm{T}}(t) e(t) \le 2V_1(0) E_{\alpha}(-2t^{\alpha}). \quad (31)$$

Moreover, we should also note the fact that, for $0 < \alpha < 1$ and $k > 0$, $E_{\alpha}(-kt^{\alpha})$ is completely monotonic and decreases much faster than the exponential function $\exp^{-kt}$(see [42]). Therefore, we can conclude from inequality (31) that $\lim_{t\to+\infty} e^{\mathrm{T}}(t) e(t) = 0$, that is, $\lim_{t\to+\infty} \|e(t)\|_2 = 0$. It means that the network (9) is synchronized under the adaptive law (13). The convergence of error vector implies, from (13) and from the fact that A is positive definite, $\lim\limits_{t\to+\infty} \mathrm{D}_t^{\alpha} G_{ij}(t) = 0$. According to Property 3, one can conclude that $G_{ij}(t)((i,j)\in E)$ converges to a finite constant. The proof is completed. ■

Remark 1. In recent years, many kinds of adaptive strategies were designed to adjust the gains of feedback controllers, see [27-29] and relevant references therein. Actually, a diffusively coupled network could be synchronized by designing suitable coupling gains among the network nodes. As a natural extension of the existing network models and control methods, a new fractional-order complex dynamical network with time-varying diffusive coupling is proposed, and then the fractional-order decentralized adaptive strategy to tune the coupling gains between the network nodes is designed based on the local mismatch between neighboring nodes. To our knowledge, this is the first paper to consider the synchronization of fractional-order complex dynamical networks with adaptive coupling. Fortunately, this challenging problem has been solved by fractional Lyapunov functional method combined with Mittag-Leffler function, Laplace transform, and fractional inequality techniques.

Remark 2. From (13), we have $\mathrm{D}_t^{\alpha}G_{ij}(t) \geq 0$. However, one cannot conclude that $G_{ij}(t)$ is monotonously non-decreasing for $0 < \alpha < 1$. To state the reason, we assume $x(t) \in C^1[t_0, +\infty)$ and satisfies

$$\mathrm{D}_t^{\alpha}x(t) = f(t,x) \geq 0,\ 0 < \alpha < 1. \tag{32}$$

$\forall t_0 \leq t_2 < t_1 < +\infty$, integrating both sides of (32) from t_0 to t_1 and t_0 to t_2 respectively, it follows from Definition 1 and Property 2 that

$$x(t_1) - x(t_0) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\frac{f(\tau,x(\tau))}{(t_1-\tau)^{1-\alpha}}\mathrm{d}\tau. \tag{33}$$

$$x(t_2) - x(t_0) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2}\frac{f(\tau,x(\tau))}{(t_2-\tau)^{1-\alpha}}\mathrm{d}\tau. \tag{34}$$

Subtracting (34) from (33), we have

$$\begin{aligned} x(t_1) - x(t_2) = &\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\frac{f(\tau,x(\tau))}{(t_1-\tau)^{1-\alpha}}\mathrm{d}\tau \\ &- \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2}\frac{f(\tau,x(\tau))}{(t_2-\tau)^{1-\alpha}}\mathrm{d}\tau \\ = &\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2}\frac{f(\tau,x(\tau))}{(t_1-\tau)^{1-\alpha}}\mathrm{d}\tau + \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}\frac{f(\tau,x(\tau))}{(t_1-\tau)^{1-\alpha}}\mathrm{d}\tau \\ &- \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2}\frac{f(\tau,x(\tau))}{(t_2-\tau)^{1-\alpha}}\mathrm{d}\tau \\ = &\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2}\left[\frac{f(\tau,x(\tau))}{(t_1-\tau)^{1-\alpha}} - \frac{f(\tau,x(\tau))}{(t_2-\tau)^{1-\alpha}}\right]\mathrm{d}\tau \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}\frac{f(\tau,x(\tau))}{(t_1-\tau)^{1-\alpha}}\mathrm{d}\tau, \end{aligned} \tag{35}$$

where $\frac{1}{(t_1-\tau)^{1-\alpha}} - \frac{1}{(t_2-\tau)^{1-\alpha}} < 0$ for $0 < \alpha < 1$. Thus, as can be seen from (35), one cannot establish the sign of $x(t_1) - x(t_2)$, which is closely related to α. Obviously, this analysis result is not consistent with that of integer-order case. It should be noted that our numerical results for coupling gains can be theoretically interpreted by the analysis result in this remark.

B. Fractional-order decentralized adaptive pinning strategy for the synchronization

In Theorem 1, all the coupling gains are adjusted according to the adaptive law (13). Here, only a small fraction of the coupling gains is updated to reach synchronization.

Let $\tilde{E}$ be a subset of E. Assume that network (9) is connected through the pinning edges $\tilde{E}$.

Here, we define

$$L_{ij} = \begin{cases} G_{ij}(0), & \text{if } (i,j) \in E - \tilde{E} \\ -\sum_{j=1,j\neq i}^{N} G_{ij}(0), & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \tag{36}$$

Theorem 2. Suppose that Assumption 1 holds. Then, the network (9) is synchronized under the following fractional-order decentralized adaptive pinning strategy:

$$\begin{aligned} &\mathrm{D}_t^{\alpha}G_{ij}(t) = \gamma_{ij}(x_i(t) - x_j(t))^{\mathrm{T}}A(x_i(t) - x_j(t)), \\ &G_{ij}(0) = G_{ji}(0) > 0,\ (i,j) \in \tilde{E}, \end{aligned} \tag{37}$$

where $\gamma_{ij} = \gamma_{ji}$ are positive constants.

Proof. Consider the following Lyapunov functional candidate for system (12)

$$V_2(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) + \sum_{i=1}^{N}\sum_{(i,j)\in\tilde{E}}\frac{c}{4\gamma_{ij}}(G_{ij}(t) - \tilde{h}_{ij})^2, \tag{38}$$

where $\tilde{h}_{ij}$ is defined as

$$\begin{aligned} &\tilde{h}_{ij} = \tilde{h}_{ji} > 0, \quad \text{if } (i,j) \in \tilde{E}, \\ &\tilde{h}_{ij} = 0 (i \neq j), \quad \text{otherwise}. \end{aligned} \tag{39}$$

Let $\tilde{H} = (\tilde{h}_{ij})_{N\times N}$, $\tilde{h}_{ii} = -\sum_{j=1,j\neq i}^{N}\tilde{h}_{ij}$. Now, we calculate the fractional derivative of V_2 along the trajectories of system (12)

$$\begin{aligned} \mathrm{D}_t^{\alpha}V_2 \leq &\sum_{i=1}^{N}\sum_{(i,j)\in\tilde{E}}\frac{c}{2\gamma_{ij}}\big(G_{ij}(t) - \tilde{h}_{ij}\big)\mathrm{D}^{\alpha}\big(G_{ij}(t) - \tilde{h}_{ij}\big) \\ &+ \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\mathrm{D}^{\alpha}e_i(t) \\ = &\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\Bigg[f(t,x_i(t)) - f(t,\bar{x}) + f(t,\bar{x}) \\ &- \frac{1}{N}\sum_{j=1}^{N} f(t,x_j(t)) + c\sum_{j=1}^{N} G_{ij}(t)Ae_j(t)\Bigg] \\ &+ \sum_{i=1}^{N}\sum_{(i,j)\in\tilde{E}}\frac{c}{2}(G_{ij}(t) - \tilde{h}_{ij})(e_i(t) - e_j(t))^{\mathrm{T}} \\ &\times A(e_i(t) - e_j(t)) \\ \leq &\ \varepsilon\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) + c\sum_{i=1}^{N}\sum_{j=1}^{N} L_{ij}e_i^{\mathrm{T}}(t)Ae_j(t) \\ &+ c\sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{h}_{ij}e_i^{\mathrm{T}}(t)Ae_j(t) \\ = &\ e^{\mathrm{T}}(t)\big[\varepsilon(I_N\otimes I_n) + c(L\otimes A) + c(\tilde{H}\otimes A)\big]e(t), \end{aligned} \tag{40}$$

where $e(t) = (e_1^{\mathrm{T}}(t), e_2^{\mathrm{T}}(t), \cdots, e_N^{\mathrm{T}}(t))^{\mathrm{T}}$, $L = (L_{ij})_{N\times N}$, $\tilde{H} = (\tilde{H}_{ij})_{N\times N}$. Then, following similar steps as in the proof of Theorem 1, we can complete the proof. ■

IV. Numerical examples

In this section, two numerical examples are given to validate the above obtained theoretical results. Here, the predictor-corrector method studied in [44] is utilized to solve the differential equations of the fractional-order systems. In the following examples, the simulation step-size is chosen as h=0.01.

Example 1. Consider a diffusively coupled scale-free[1] network with 50 nodes, where each node is a fractional-order non-autonomous parametrically excited Duffing oscillator described by

$$\begin{aligned} &\mathrm{D}_t^{\alpha} x_1 = x_2, \\ &\mathrm{D}_t^{\alpha} x_2 = (1+\mu\sin(\omega t))x_1 - \gamma x_2 - x_1^3. \end{aligned} \tag{41}$$

When $\mu = 0.5$, $\omega = 1$, $\gamma = 0.2$, $\alpha = 0.975$, system (41) has a chaotic attractor as shown in Fig.1.

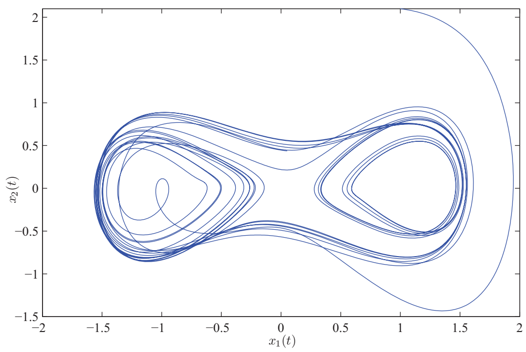

Fig. 1. (color online) Chaotic attractor of system (41) with $\mu = 0.5$, $\omega = 1$, $\gamma = 0.2$, $\alpha = 0.975$ and $(x_1(0), x_2(0)) = (1.0,\ 2.1)$

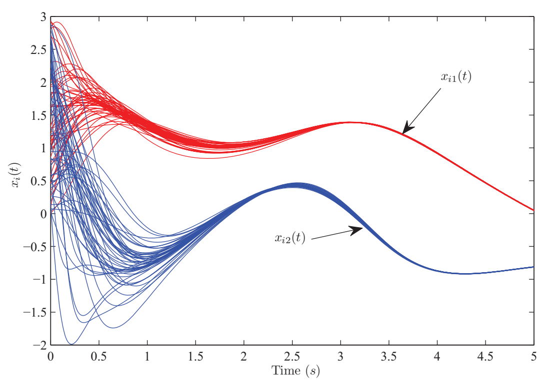

Fig. 2. (color online) Time evolutions of $x_i = (x_{i1}, x_{i2})^{\mathrm{T}}, i = 1, 2, \cdots, 50$

For simplicity and without losing generality, we take $c = 1$, $A = \mathrm{diag}(1, 1, 1)$. For connected nodes i and j, $G_{ij}(0) = G_{ji}(0)$ are chosen randomly in $(0,\ 1)$ and $\gamma_{ij} = \gamma_{ji} = 1$, $\forall (i, j) \in E$. The initial states x_i are chosen randomly in $(0,\ 3)$. Therefore, all the conditions of Theorem 1 are satisfied, and the network synchronization is asymptotically achieved. As shown in Figs.2 and 3, the simulation results agree well with the theoretical analysis.

Example 2. Consider a diffusively coupled complex network with 10 nodes, where each node is a fractional-order Arneodo's system described by

$$\begin{aligned} &\mathrm{D}_t^{\alpha} x_1 = x_2, \\ &\mathrm{D}_t^{\alpha} x_2 = x_3 \\ &\mathrm{D}_t^{\alpha} x_3 = \beta_1 x_1 - \beta_2 x_2 - \beta_3 x_3 + \beta_4 x_1^3. \end{aligned} \tag{42}$$

When $\beta_1 = 5.5$, $\beta_2 = 3.5$, $\beta_3 = 0.4$, $\beta_4 = -1$, and $\alpha = 0.9$, system (42) is chaotic[45]. We take $c = 1$,

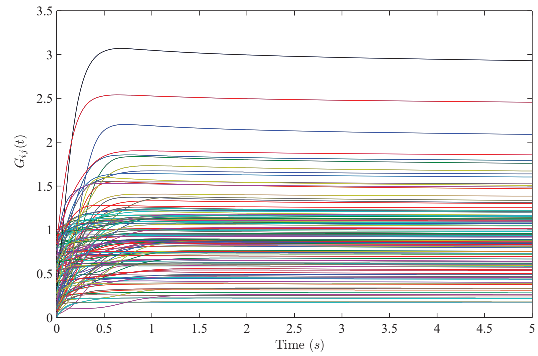

Fig. 3. (color online) Adaptive coupling gains $G_{ij}(t),\ (i, j) \in E$

$A = \mathrm{diag}(1, 1, 1)$. The initial coupling matrix is chosen as

$$\mathrm{G}(0) = \begin{bmatrix} -3.8 & 0.8 & 0.6 & 0.3 & 0 & 0.3 & 0.6 & 0.7 & 0.5 & 0 \\ 0.8 & -2.5 & 0.2 & 0 & 0.4 & 0 & 0 & 0.2 & 0.2 & 0.7 \\ 0.6 & 0.2 & -1.8 & 0.4 & 0 & 0.5 & 0.1 & 0 & 0 & 0 \\ 0.3 & 0 & 0.4 & -0.8 & 0.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0.1 & -0.6 & 0 & 0 & 0 & 0 & 0.1 \\ 0.3 & 0 & 0.5 & 0 & 0 & -1.1 & 0.3 & 0 & 0 & 0 \\ 0.6 & 0 & 0.1 & 0 & 0 & 0.3 & -1.2 & 0 & 0.2 & 0 \\ 0.7 & 0.2 & 0 & 0 & 0 & 0 & 0 & -0.9 & 0 & 0 \\ 0.5 & 0.2 & 0 & 0 & 0 & 0 & 0.2 & 0 & -0.9 & 0 \\ 0 & 0.7 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0 & -0.8 \end{bmatrix}$$

Here, we select a fraction of coupling gains in the network. Choose $\gamma_{12} = \gamma_{21} = 0.5$, $\gamma_{13} = \gamma_{31} = 0.6$, $\gamma_{17} = \gamma_{71} = 0.7$, $\gamma_{18} = \gamma_{81} = 0.8$, $\gamma_{19} = \gamma_{91} = 0.9$, $\gamma_{25} = \gamma_{52} = 0.5$, $\gamma_{2,10} = \gamma_{10,2} = 0.6$, $\gamma_{36} = \gamma_{63} = 0.7$, $\gamma_{34} = \gamma_{43} = 0.8$. The initial states x_i are chosen randomly in $(0,\ 2)$. According to Theorem 2, the network synchronization is asymptotically achieved. The simulation results depicted in Figs.4 and 5 agree well with the theoretical analysis. As can be seen from Figs.3 and 5, the adaptive coupling gains are not monotonously non-decreasing, which further validates our theoretical analysis in Remark 2.

V. Conclusions

In this paper, two fractional-order decentralized adaptive strategies have been proposed to tune the coupling gains between network nodes. Based on the proposed adaptive coupling strategies, two sufficient conditions have been derived for synchronization of fractional-order complex networks. In the proofs of the theorems, an inequality has been used to estimate the fractional-order derivative of a quadratic Lyapunov function. Thus, we can investigate the synchronization for

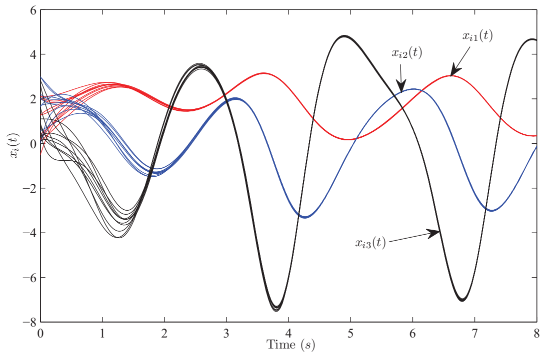

Fig. 4. (color online) Time evolutions of $x_i = (x_{i1}, x_{i2}, x_{i3})^{\mathrm{T}}$, $i = 1, 2, \cdots, 10$.

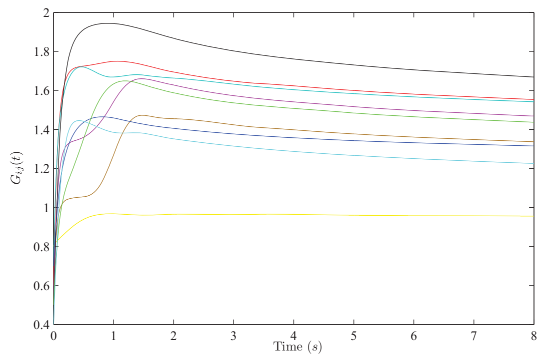

Fig. 5. (color online) Adaptive coupling gains $G_{ij}(t), (i, j) \in \tilde{E}$.

fractional-order complex networks like integer-order complex networks. Numerical examples have been given to validate the theoretical results. The obtained results show that the adaptive coupling gains are not monotonously non-decreasing even though $\mathrm{D}_t^{\alpha} G_{ij}(t) \geq 0$. This counter-intuitive conclusion also implies that the fractional-order system has additional attractive feature over the integer-order system.

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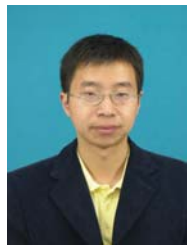
Quan Xu graduated from Xihua University, China, 2006. He received the M.Sc. degree from Xihua University, China, in 2009. He is currently a Lecturer with Xihua University, and he is also working towards the Ph.D. degree at the School of Electrical Engineering, Southwest Jiaotong University, China. His research interests include nonlinear control, fractional-order systems, stability theory, complex networks. He serves as a reviewer for several journals.

Shengxian Zhuang was born in Hunan, China, in 1964. He received his M.Sc. degree from Southwest Jiaotong University, and Ph.D. degree from University of Electronics and Technology of China, in 1991 and 1999 respectively. He has been a Professor of Electrical Engineering at the Southwest Jiaotong University since 2003. His research interests include nonlinear control, stability theory and power electronics.



Yingfeng Zeng graduated from Xihua University, 2001. He received his M.Sc. degree from Xihua University, in 2009. He is working towards the Ph.D. degree at the School of Electrical Engineering, Southwest Jiaotong University, China. His research interest covers the controllability and stability of complex networks.

Jian Xiao graduated from Hunan University, 1979. He received his M.Sc. degree from Hunan University, and Ph.D. degree from Southwest Jiaotong University, in 1982 and 1989 respectively. He has been a Professor of Electrical Engineering at the Southwest Jiaotong University since 1994. His research interests include robust control, fuzzy systems and power electronics.