Fuzzy Adaptive Control of a Fractional Order Chaotic System with Unknown Control Gain Sign Using a Fractional Order Nussbaum Gain

Khatir Khettab, Samir Ladaci, *Member, IEEE*, and Yassine Bensafia

***Abstract*—In this paper we propose an improved fuzzy adaptive control strategy, for a class of nonlinear chaotic fractional order (SISO) systems with unknown control gain sign. The online control algorithm uses fuzzy logic sets for the identification of the fractional order chaotic system, whereas the lack of a priori knowledge on the control directions is solved by introducing a fractional order Nussbaum gain. Based on Lyapunov stability theorem, stability analysis is performed for the proposed control method for an acceptable synchronization error level. In this work, the Grünwald-Letnikov method is used for numerical approximation of the fractional order systems. A simulation example is given to illustrate the effectiveness of the proposed control scheme.**

***Index Terms*—Adaptive fuzzy control, nonlinear fractional order systems, fractional order Nussbaum function, chaos synchronization, Lyapunov stability.**

I. Introduction

FHERE order chaotic systems are gathering an important research effort because of their powerful properties and potential applications in secure communication and control processing. Many mathematical models have been developed in literature such as the fractional-order Chua system[1], the fractional-order Duffing system[2], the fractional-order Lü system and the fractional order Chen system[3]. Since the work of Deng and Li[4] who investigated the synchronization problem of fractional order chaotic Lü systems, many studies were focused on the control and synchronization of fractional order chaos[5−6].

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In this paper we are concerned by fractional adaptive control of nonlinear fractional order systems using Fuzzy logic identification technique. Fractional adaptive systems have been largely investigated for a decade as they showed an improved behavior comparatively to classical adaptive control for partially unknown plants [7−10].

Based on the universal approximation theorem, adaptive fuzzy control systems present an effective control solution for a large class of nonlinear systems[11]. The adaptive controller is synthesized from a collection of fuzzy IF-THEN rules and the parameters of the membership functions characterizing the linguistic terms in the IF-THEN rules change according to some adaptive law for the purpose of controlling a plant to track a reference trajectory[12−13].

A particular class of such nonlinear plants pose the challenging control problem with unknown control directions[15]. The Nussbaum function approach was introduced in the 1980's[16]. This technique was used for adaptive control of first-order nonlinear systems in[17]. Later, many studies of adaptive control schemes with Nussbaum function were successfully carried out for different classes of nonlinear systems[18−20].

The main contribution of this study is the introduction of a Nussbaum function in the fuzzy adaptive control scheme for nonlinear fractional systems with unknown control gain sign. Stability analysis of the proposed adaptive fuzzy control system is performed using Lyapunov stability theory. Moreover, the influence of the approximation error and external disturbance on the tracking error can be attenuated to an arbitrarily prescribed level via the proposed design technique. The fuzzy adaptive control design with Nussbaum function is applied for nonlinear fractional order chaotic systems with a large uncertainty or unknown variation in plant parameters and structures. The Grünwald-Letnikov technique is used for the numerical approximation of the fractional order chaotic system[21].

This paper is organized as follows: in Section II, basic definitions and preliminaries for fractional order systems are presented with the numerical approximation technique. A description of the Nussbaum-type function is given in Section III. Section IV presents the fuzzy adaptive control scheme with unknown control direction for uncertain fractional order chaotic systems in the presence of uncertainty.

The proposed control system stability proof is detailed in Section V. In Section VI, application of the proposed method on fractional order chaotic Duffing systems is investigated. Finally, the simulation results and conclusion are presented in Section VII.

II. Basics of fractional order systems

A. Fractional Derivatives and Integrals

The mathematical definition of fractional derivatives and integrals has been the subject of several descriptions. The three most frequently used definitions for the general fractional differ-integral are: the Grünwald-Letnikov (GL) definition, the Riemann-Liouville (RL) and the Caputo definition[9],[21]. The Riemann-Liouville (RL) definition of the fractional order integral is given by:

$$ {}_a^{RL}D_t^{-\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t-\tau)^{\mu-1} f(\tau) d(\tau), \tag{1} $$

while the definition of fractional order derivatives is

$$ \begin{aligned} {}_a^{RL}D_t^{\mu} f(t) &= \frac{d}{dt}\left[{}_a^{RL}D_t^{-(1-\mu)} f(t)\right], \\ &= \frac{1}{\Gamma(1-\mu)}\frac{d}{dt}\int_a^t (t-\tau)^{-\mu} f(\tau) d(\tau), \end{aligned} \tag{2} $$

where

$$ \Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy, \tag{3} $$

where $\Gamma(.)$ is the Euler's Gamma function, a and t are the limits of the operation, and μ is the number identifying the fractional order. In this paper, μ is assumed to be a real number that satisfies the restriction $0 < \mu < 1$. Also, it is assumed that $a = 0$. The following convention is used: ${}_aD_t^{\mu} \equiv D^{\mu}$.

B. Numerical Approximation Method

Many different approaches have been proposed to model fractional order systems. The numerical simulation of such systems depends on the way to approximate the fractional derivative operator. The most common approach used in the fractional order chaotic systems literature is an improved version of the Adams-Bashforth-Moulton method based on predictor-correctors[22−23]. However, we will use in this work a simpler approach consisting of the fractional order derivative operator discretization according to the Grünwald-Letnikov method. This method is very simple to use and has approximately the same order of accuracy as the predictor-corrector method, even if the simulation requires, for each step the computation of sums of increasing dimension with time.

The Grünwald-Letnikov fractional order derivative definition is expressed as:

$$ {}_a^{GL}D_t^{-\mu} = \lim_{h\to 0}\frac{1}{h^{\mu}} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \begin{pmatrix} \mu \\ j \end{pmatrix} f(t-jh), \tag{4} $$

where $\left[\frac{t-a}{h}\right]$ indicates the integer part and $(-1)^j \begin{pmatrix} \mu \\ j \end{pmatrix}$ are binomial coefficients $c_j^{(\mu)}$, $(j = 0, 1, \ldots)$.

The calculation of these coefficients is done by a formula of following recurrence:

$$ c_0^{(\mu)} = 1;\; c_j^{(\mu)} = \left(1 - \frac{1+\mu}{j}\right) c_{j-1}^{(\mu)}. $$

The general numerical solution of the fractional differential equation,

$$ {}_a^{GL}D_t^{-\mu} = f\left(y(t)\right), \tag{5} $$

can be expressed as follows:

$$ y(t_k) = f\left(y(t_k), t_k\right) h^{\mu} - \sum_{j=0}^{k} c_j^{(\mu)} y(t_{k-j}). \tag{6} $$

This approximation of the fractional derivative within the meaning of Grünwald-Letnikov is on the one hand equivalent to the definition of Riemman-Liouville for a broad class of functions[24], and on the other hand, it is well adapted to the definition of Caputo (Adams method) because it requires only the initial conditions and has a clear physical direction.

III. Nussbaum-type gain

Definition 1: A function $N(\zeta)$ is called a Nussbaum-type function if it has the following properties[14],[25−27]:

$$ \lim_{s\to\infty} \sup \frac{1}{s}\int_0^s N(\zeta) d\zeta = +\infty, \tag{7} $$

$$ \lim_{s\to\infty} \inf \frac{1}{s}\int_0^s N(\zeta) d\zeta = -\infty. \tag{8} $$

The continuous functions $N_1(\zeta) = \zeta^2\cos(\zeta)$, $N_2(\zeta) = \zeta\cos(\sqrt{|\zeta|})$, $N_3(\zeta) = \cos(\frac{\pi}{2}\zeta)e^{\zeta^2}$ and $N_4(\zeta) = \ln(\zeta + 1)\cos(\sqrt{\ln(\zeta+1)})$ are Nussbaum functions.
For example the continuous function $N_1(\zeta) = \zeta^2\cos(\zeta)$, is positive at interval $(2\pi n, 2\pi n + \frac{\pi}{2})$ and negative at the interval $(2\pi n + \frac{\pi}{2}, 2\pi n + \frac{3\pi}{2})$, where n is an integer. And we have,

$$ \frac{1}{2\pi n + \frac{\pi}{2}} \int_0^{2\pi n + \frac{\pi}{2}} N_1(\zeta) d\zeta = +\infty, $$

$$ \frac{1}{2\pi n + \frac{3\pi}{2}} \int_0^{2\pi n + \frac{3\pi}{2}} N_1(\zeta) d\zeta = -\infty. $$

The following lemma[28] is used in the stability analysis.

Lemma 1: Consider the following fractional-order system,

$$ D^{\alpha} y(t) = -a y(t) + b. \tag{9} $$

then there exists a constant $t_0 > 0$ such that for all $t \in (t_0, \infty)$,

$$ \|y(t)\| \le \frac{2b}{a}, \tag{10} $$

where $y(t)$ is the state variable, and a, b are two positive constants.

The proof of Lemma 1 can be found in[28].
A Nussbaum function will be used in future work, to estimate the control direction.

IV. PROBLEM STATEMENT

Consider a fractional order SISO nonlinear dynamic system of the form:

$$\begin{cases} x_1^{(q_1)} = x_2, \\ \vdots, \\ x_{n-1}^{(q_{n-1})} = x_n, \\ x_n^{(q_n)} = f(\underline{x}, t) + g(\underline{x}, t)u + d(t), \\ y = x_1, \end{cases} \tag{11}$$

where,
$\underline{x} = [x_1, x_2, ..., x_n]^{\mathrm{T}} = [x, x^{(q)}, x^{(2q)}, ..., x^{((n-1)q)}]^{\mathrm{T}} \in \mathbf{R}^n$ is the system's state vector, $u \in \mathbf{R}$ is the control input and $y \in \mathbf{R}$ is the output, with the initial conditions : $u(0) = 0$ and $y(0) = 0$.
The initial conditions are set to zero to avoid the lack of robustness for Nussbaum type adaptive controllers as proved by Georgiou and Smith[29],
If $q_1 = q_2 = ... = q_n = q$ the above system is called a commensurate order system. Then an equivalent form of the above system is described as :

$$\begin{cases} x^{(nq)} = f(\underline{x}, t) + g(\underline{x}, t)u + d(t), \\ y = x_1, \end{cases} \tag{12}$$

where $f(\underline{x}, t)$ and $g(\underline{x}, t)$ are unknown but bounded nonlinear functions which express system dynamics and $d(t)$ is the external bounded disturbance. The control objective is to force the system output $\underline{y}$ to follow a given bounded reference signal $\underline{y}_d$, under the constraint that all signals involved must be bounded.
The reference signal vector $\underline{y}_d$ and the tracking error vector $\underline{e}$ are defined as,

$$\begin{aligned} \underline{y}_d &= [y_d,\, y_d^{(q)},\, y_d^{(2q)},\, ...,\, y_d^{((n-1)q)}]^{\mathrm{T}} \in \mathbf{R}^n, \\ \underline{e} &= \underline{y}_d - \underline{y} = [e,\, e^{(q)},\, ...,\, e^{((n-1)q)}]^{\mathrm{T}} \in \mathbf{R}^n, \\ e^{(iq)} &= y_d^{(iq)} - y^{(iq)}. \end{aligned}$$

Let $\underline{k} = [k_1,\, k_2,\, ...,\, k_n]^{\mathrm{T}} \in \mathbf{R}^n$ be chosen such that the stability condition $|\arg(eig(A))| > q\pi/2$ is met, where $0 < q < 1$ and $eig(A)$ represents the eigenvalues of the system state matrix.

i) Let us first suppose that the functions $f(\underline{x}, t)$ and $g(\underline{x}, t)$ are known and the system is free of external disturbance (i.e. $d(t) = 0$).
The following assumptions are considered[19−20],

Assumption 1: The control gain $g(\underline{x}, t)$ is not zero and of known sign. It is also strictly positive or strictly negative.

Assumption 2: The external disturbance is bounded: $|d(t)| \leq D$ with D an unknown positive constant.

Then the control law of the certainty equivalent controller is obtained as[30],

$$u^* = \frac{1}{g(\underline{x}, t)} \left(-f(\underline{x}, t) + y_d^{(nq)} + k^T e \right), \tag{13}$$

where

$$\underline{y}_d = [y_d,\, y_d^{(q)},\, y_d^{(2q)},\, ...,\, y_d^{((n-1)q)}]^{\mathrm{T}} \in \mathbf{R}^n,$$

$$\begin{aligned} \underline{e} &= \underline{y}_d - \underline{y} = [e,\, e^{(q)},\, ...,\, e^{((n-1)q)}]^{\mathrm{T}} \in \mathbf{R}^n, \\ e^{(iq)} &= y_d^{(iq)} - y^{(iq)}, \end{aligned}$$

is the tracking error vector.
Substituting (13) into (12), we have:

$$e^{nq} = k_n e^{(n-1)q} + \cdots + k_1 e = 0, \tag{14}$$

which is the main objective of control, $\lim_{t\to\infty} e(t) = 0$.

ii) However, $f(\underline{x}, t)$ and $g(\underline{x}, t)$ are unknown and external disturbance $d(t) \neq 0$, the ideal control effort (13) cannot be implemented; this problem was solved by the control strategy proposed previously by the use of fuzzy systems to approximate unknown functions[13]. In this case, we consider the following assumptions[19−20]:

Assumption 3: The state vector x is not measurable, except the system output y.

Assumption 4: The reference trajectory $y_d(t)$ and its derivatives up to order (nq) are known, continuous and bounded.

Assumption 5: The control gain $g(\underline{x}, t)$ is not zero and of unknown sign.

Remark 1: In Assumption 5, and contrary to the previous case, the sign of $g(\underline{x}, t)$ need not to be known, as the *Nussbaum* technique will estimate the control gain sign.

From Definition 1, one knows that Nussbaum functions should have infinite gains and infinite switching frequencies. Subsequently to this part, the Nussbaum function

$$N(\zeta) = \zeta^2 \cos(\zeta),$$

will be used for the control of nonlinear chaotic systems.
By substituting (13) into (12) we obtain the closed loop control system in the state space domain as follows:

$$\begin{aligned} \underline{x}^{(nq)} &= A\underline{x} + B[f(\underline{x}) + g(\underline{x})u], \\ y &= c^T \underline{x}, \end{aligned} \tag{15}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -k_1 & -k_2 & -k_3 & -k_4 & \cdots & -k_{(n-1)} & -k_n \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \text{ and, } \quad c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

By using the relation $y_d^{(q)} = Ay_d + By_d^{(nq)}$ the following equation (16) is obtained:

$$\begin{aligned} \underline{e}^{(q)} &= A\underline{e} + B[f(\underline{x}) + g(\underline{x})u^* - y_d^{(nq)}], \\ e &= c^T \underline{e}. \end{aligned} \tag{16}$$

In what follows, a fuzzy adaptive control will be designed to stabilize the system (11) or the equivalent system (16).
Replacing $f(\underline{x})$ by the fuzzy system $f(\underline{x}, \theta_f)$ which is specified as:

$$f(\underline{x}, \theta_f) = \theta_f^T \xi(\underline{x}). \tag{17}$$

Here the fuzzy basis function $\xi(\underline{x})$ depends on the fuzzy membership functions and is supposed to be fixed, while θ_f is the adjusted by adaptive laws based on a Lyapunov stability criterion.
Using (17), (16) can be rewritten as following:

$$\begin{aligned} \underline{e}^{(q)} &= A\underline{e} + B[\xi^T(\underline{x})\theta_f + g(\underline{x})u^* - y_d^{(nq)}], \\ e &= c^T\underline{e}. \end{aligned} \tag{18}$$

The optimal parameter estimation vector θ_f^* is defined by:

$$\theta_f^* = \arg\min_{\theta_f \in \Omega_f} \left[\sup_{x \in \Omega_x} |f(\underline{x} \mid \theta_f) - f(\underline{x}, t)| \right]. \tag{19}$$

with $\phi_f = \theta_f - \theta_f^*$ and Ω_f, Ω_x are constraints sets for θ_f and x respectively, and are defined as:

$$\begin{aligned} \Omega_f &= \{\theta_f \mid \ |\theta_f| \leq M_f\}, \\ \Omega_x &= \{x \mid \ |x| \leq M_x\}, \end{aligned} \tag{20}$$

where M_f and M_x are positive constants.
The following theorem is proposed to show the control performance of the closed loop system.

Theorem 1: Considering system (12), and the fuzzy adaptive control law proposed with fractional Nussbaum function is given as follows:

$$u^* = N(\zeta)\left[k^T\underline{e} + \theta_f^T\xi(\underline{x}) - y_d^{nq}\right], \tag{21}$$

where $N(\zeta) = \zeta^2\cos(\zeta)$ and,

$$\zeta^{(q)} = \underline{e}^T PB\left[k^T\underline{e} + r_1\theta^T\xi(\underline{x}) - y_d^{(nq)}\right], \tag{22}$$

and the fractional adaptive law for the vector θ is chosen as following:

$$\theta^{(q)} = -r_1\theta + r_1\underline{e}^T PB\xi(\underline{x}), \tag{23}$$

where r_1 is a positive constant, and $P = P^T > 0$ is a positive definite matrix, also there is a positive definite symmetric matrix $Q = Q^T$ satisfying the following Lyapunov equation:

$$A_c P + PA_c^T + PBB^T P = -Q.$$

We choose $A_c = A - B\underline{k}^T$ is Hurwitz. So all signals in the closed loop system are bounded and the tracking error converges to a bounded compact set defined by $\Omega = \{e_1, |e_1| \leq a_1\}$, where a_1 is a positive constant.

V. Stability Analysis

The Lyapunov function is chosen as

$$V = \frac{1}{2}\underline{e}^T P\underline{e} + \frac{1}{2r_1}\phi_f^T\phi_f. \tag{24}$$

The derivative of (24) with respect to time verifies[31−32]:

$$V^{(q)}(t) \leq \frac{1}{2}(\underline{e}^{(q)})^T P\underline{e} + \frac{1}{2}\underline{e}^T(t)P\underline{e}^{(q)}(t) + \frac{1}{r_1}\phi_f^T\phi_f^{(q)}. \tag{25}$$

By substituting (18) into (25), we obtain:

$$\begin{aligned} V^{(q)}(t) \leq{} & \frac{1}{2}\underline{e}^T\left(PA + A^T P\right)\underline{e} + \frac{1}{r_1}\phi_f^T\phi_f^{(q)} \\ & + \underline{e}^T PB\left[\xi(\underline{x})\theta_f^T + gu^* - y_d^{(nq)}\right]. \end{aligned} \tag{26}$$

By using (21) and (22), (26) becomes:

$$\begin{aligned} V^{(q)}(t) \leq{} & \frac{1}{2}\underline{e}^T\left(PA + A^T P\right)\underline{e} + \frac{1}{r_1}\phi_f^T\phi_f^{(q)} \\ & + \underline{e}^T PB\left[\xi(\underline{x})\theta_f^T + gu^* - y_d^{(nq)}\right] - \underline{e}PB\theta_f^T\xi(\underline{x}) \\ \leq{} & \frac{1}{2}\underline{e}^T\left(PA + A^T P\right)\underline{e} + \phi_f^T\left[\frac{1}{r_1}\phi_f^{(q)} - \underline{e}^T PB\xi(\underline{x})\right] \\ & + \underline{e}^T PB\left[\xi(\underline{x})\theta_f^T - y_d^{(nq)}\right] + \underline{e}^T PBgN(\zeta)\left(k^T\underline{e}\right. \\ & \left. + \xi(\underline{x})\theta_f^T - y_d^{(nq)}\right) \\ \leq{} & \frac{1}{2}\underline{e}^T\left(PA_c + A_c^T P\right)\underline{e} + \phi_f^T\left[\frac{1}{r_1}\phi_f^{(q)} - \underline{e}^T PB\xi(\underline{x})\right] \\ & + \left[gN(\zeta) + 1\right]\zeta^{(q)}. \end{aligned} \tag{27}$$

Using (23), the following inequality is obtained

$$\begin{aligned} \phi_f^T\left[\frac{1}{r_1}\phi_f^{(q)} - \underline{e}^T PB\xi(\underline{x})\right] &= -\phi_f^T\theta = -\phi_f^T\phi_f - \phi_f^T\theta_f^* \\ &\leq -\frac{1}{2}\phi_f^T\phi_f + \frac{1}{2}\left\|\theta_f^*\right\|^2. \end{aligned} \tag{28}$$

And thus (Young inequality),

$$\underline{e}^T PB \leq \frac{1}{2}\underline{e}^T PBB^T P\underline{e} + \frac{1}{2}b^2, \tag{29}$$

where b is a positive constant.

By Substituting (28) and (29) into (27), the following inequality is obtained:

$$\begin{aligned} V^{(q)}(t) \leq{} & \frac{1}{2}\underline{e}^T\left(PA_c + A_c^T P\right)\underline{e} + \frac{1}{2}\underline{e}^T PBB^T P\underline{e} \\ & + \frac{1}{2}b^2 - \frac{1}{2}\phi_f^T\phi_f + \frac{1}{2}\left\|\theta_f^*\right\|^2 + \left[gN(\zeta) + 1\right]\zeta^{(q)} \\ \leq{} & -\frac{1}{2}\underline{e}^T Q\underline{e} + \frac{1}{2}b^2 - \frac{1}{2}\phi_f^T\phi_f + \frac{1}{2}\left\|\theta_f^*\right\|^2 \\ & + \left[gN(\zeta) + 1\right]\zeta^{(q)} \end{aligned} \tag{30}$$

where $\mu = \lambda_{min}(QP^{-1}, r_1)$ and $\beta = \left\|\theta_f^*\right\|^2 + \frac{1}{2}b^2$.
The inequality (30) can be expressed as:

$$V^{(q)} \leq -\mu V + \omega, \tag{31}$$

where $\omega = \beta + \left[gN(\zeta) + 1\right]\zeta^{(q)}$.

Then depending on the sign of ω two cases arise:

1) If $\omega \leq 0$ then we have $V^{(q)} \leq 0$ and the uniform continuity of the fractional order derivative (3) allows to apply Barbalats lemma[33]. Hence, $V(t)$ is bounded and e and θ_f are also bounded.
2) If $\omega > 0$ then according to Lemma 1, we have

$$\|V(t)\| \leq \frac{2\omega}{\mu}. \tag{32}$$

which yields that

$$\|e(t)\| \leq 2\sqrt{\frac{\omega}{\mu\lambda_{min}(P)}}.$$

This means that $\|e(t)\|$ can be made arbitrarily small, and θ_f is bounded. From (21), u is bounded. Then all the signals in the closed loop system are bounded.

The diagram of the proposed control is given in Fig. 1.

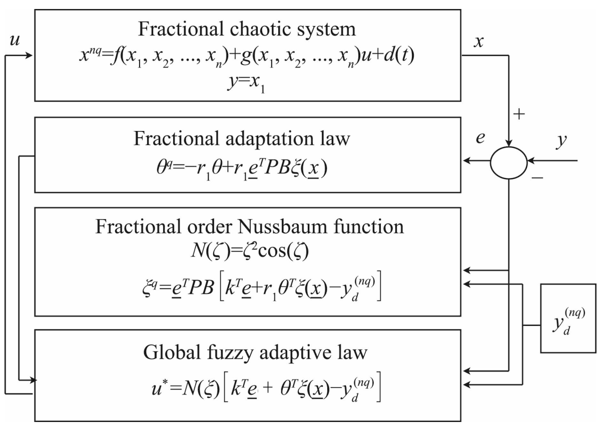

Fig. 1. Global block diagram of the proposed fuzzy adaptive control with unknown control sign gain.

VI. SIMULATION RESULTS

To illustrate the performance of the proposed control approach, we consider two fractional order chaotic systems of Duffing as follows [34],

The first one is a reference system:

$$\begin{aligned} D^q y_{d1} &= y_{d2}, \\ D^q y_{d2} &= 1.2\, y_{d1} - y_{d2} - y_{d1}^2 + 0.5\cos(t). \end{aligned} \tag{33}$$

The second is the response system (to be controlled):

$$\begin{aligned} D^q y_1 &= y_2, \\ D^q y_2 &= y_1 - 1.8\, y_2 - y_1^2 + 0.9\cos(t) + u(t) + d(t). \end{aligned} \tag{34}$$

Initial conditions are selected as follows:
$y_d(0) = [0, 0]^{\mathrm{T}}$ and $y(0) = [1, -1]^{\mathrm{T}}$.
We consider in this case the fractional order value $q = 0.98$, with the external disturbance $d(t) = 0.1\sin(t)$.

The other design constants are set as:
$k_1 = k_2 = 1$, $r_1 = 200$, $\rho = 0.05$, $h = 0.01$ and $T_{sim} = 40\ s$.

The main objective is to control our response system to track the reference system output with consideration that the functions $f(\underline{x}, t)$ and $g(\underline{x}, t)$ are completely unknown.

Fig. 2 shows the phase plane without the studied control systems.

Results & Discussion:

- According to the Fig. 4, the trajectories of the responses converge accurately to the reference trajectories, even in the presence of external disturbances.
- One can remark the vibrations in the beginning of Fig. 4(b) and Fig. 4(c). This transitory phase is necessary for converge of the system model estimated parameters. They depend mainly on the arbitrary choice of initial conditions.

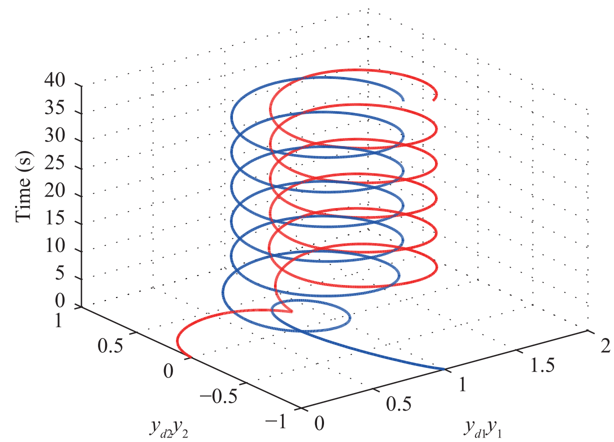

Fig. 2. Phase portrait of Duffing chaotic systems (without control action).

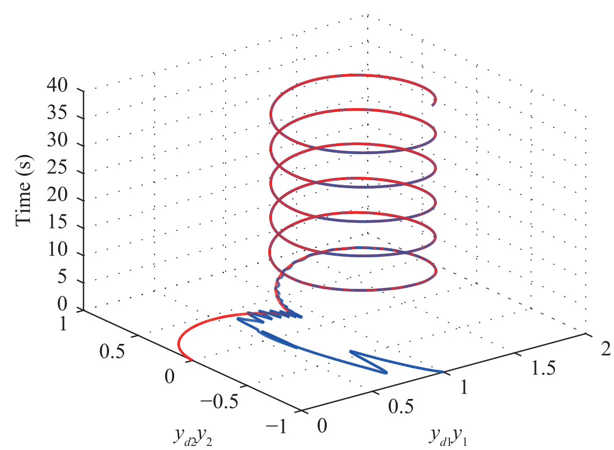

Fig. 3. Synchronization performance of Duffing chaotic drive and response systems.

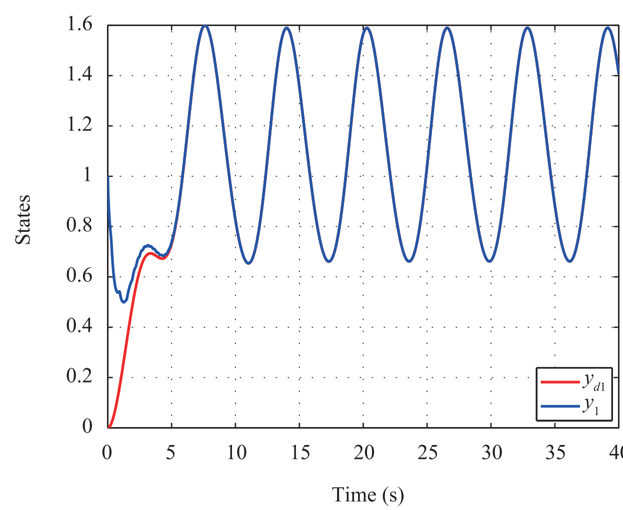

Fig. 4. (a) Trajectories of the states of systems y_1 and y_{d1}.

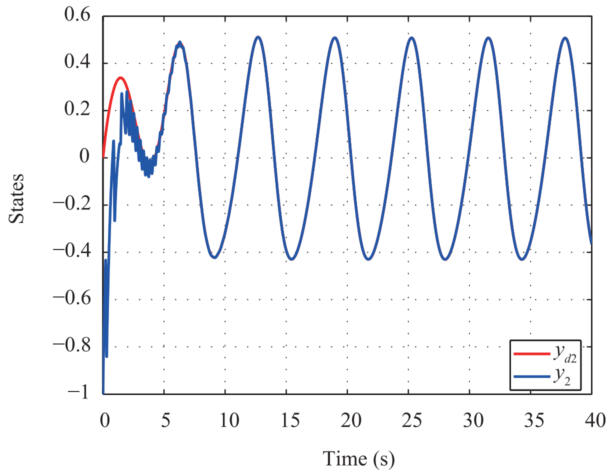

Fig. 4. (b) Trajectories of the states of systems y_2 and y_{d2}.

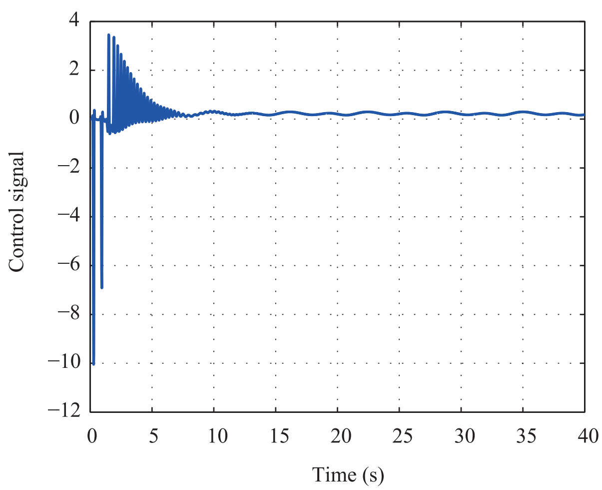

Fig. 4. (c) Control signal $u(t)$.

- Fig. 5 shows that the errors are bounded and converge asymptotically to zero.
- From Fig. 6 and Fig. 7 one can remark that the adopted settings and the function of Nussbaum which estimates the sign of control gain are always bounded.

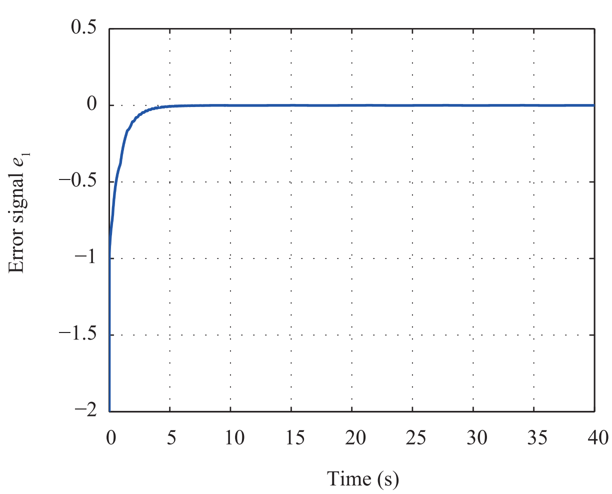

Fig. 5. (d) The error signal $e_1 = y_1 - y_{d1}$.

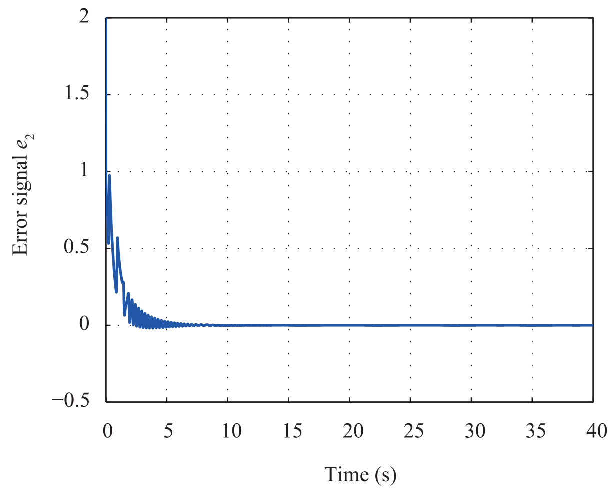

Fig. 5. (e) The error signal $e_2 = y_2 - y_{d2}$.

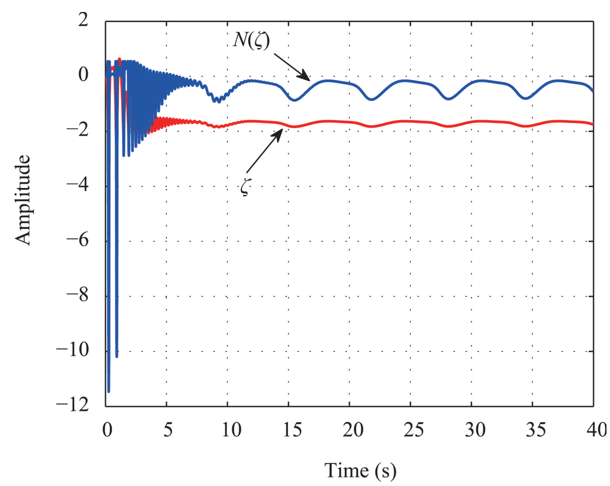

Fig. 6. Nussbaum function $N(\zeta)$ and its variation $\zeta(t)$.

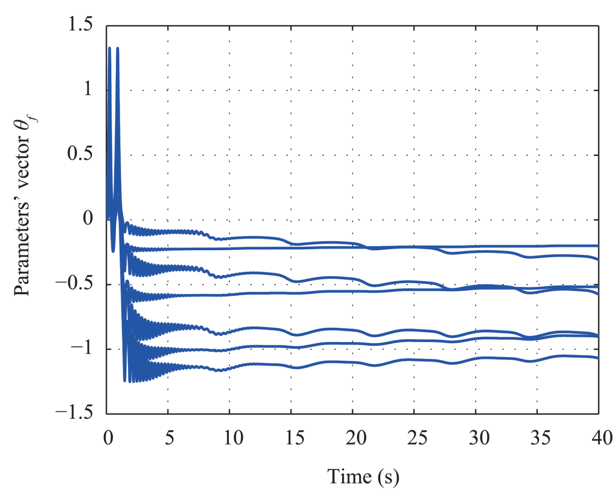

Fig. 7. Optimal parameters vector $\theta_f(t)$.

VII. Conclusion

In this work, a fuzzy adaptive control scheme is proposed for a class of nonlinear fractional order SISO systems with unknown control gain sign. The fuzzy systems were used to approximate online the unknown dynamics including all nonlinearities of the system. The numerical approximation of the fractional order systems is realized by means of the Grünwald-Letnikov method.

The main contribution of this paper is to introduce the technique of fractional order Nussbaum-type function to estimate the control gain sign for the fractional chaotic system. The developed controller guarantees the boundedness of all the signals in the closed-loop and the tracking error convergence. Simulation results show the good tracking performance of the proposed fuzzy adaptive control method.

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