Fractional-Order Control for a Novel Chaotic System without Equilibrium

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***Abstract*—The control problem is discussed for a chaotic system without equilibrium in this paper. On the basis of the linear mathematical model of the two-wheeled self-balancing robot, a novel chaotic system which has no equilibrium is proposed. The basic dynamical properties of this new system are studied via Lyapunov exponents and Poincaré map. To further demonstrate the physical realizability of the presented novel chaotic system, a chaotic circuit is designed. By using fractional-order operators, a controller is designed based on the state-feedback method. According to the Gronwall inequality, Laplace transform and Mittag-Leffler function, a new control scheme is explored for the whole closed-loop system. Under the developed control scheme, the state variables of the closed-loop system are controlled to stabilize them to zero. Finally, the numerical simulation results of the chaotic system with equilibrium and without equilibrium illustrate the effectiveness of the proposed control scheme.**

***Index Terms*—Chaotic system, Circuit implementation, Fractional-order, Stabilization.**

I. INTRODUCTION

FROM From the simplified equation of convection rolls in the equations of the atmosphere, the first three-dimensional chaotic system was derived by Lorenz in 1963[1]. With the development and applying of chaos theory, a number of chaotic systems, hyperchaotic systems, fractional-order chaotic systems and fractional-order hyperchaotic systems have been proposed, such as Rössler chaotic system[2], Liu chaotic system[3], hyperchaotic Chen system[4], hyperchaotic Lü system[5], fractional-order financial system[6], fractional-order Lotka-Volterra equation[7], fractional-order hyperchaos Lorenz system[8], a modified four-dimensional fractional order hyperchaotic system[9] and so on. The above mentioned chaotic systems have equilibrium. In addition, there are a number of chaotic systems without equilibrium which have been studied by [10-13]. As a result, chaos control became one of the important issues for chaotic systems. Due to great potential application in electrical engineering, information processing and secure communication, it is important to investigate new control methods for chaotic systems.

Over the past few decades, chaos control and chaos synchronization have received much attention and many important results have been reported. In the early 1990s, the synchronization of chaotic systems was achieved by Pecora and Carroll[14,15], which was a trailblazing result, and the result promoted the development of chaos control and chaos synchronization[16,17]. In recent years, different chaos control and chaos synchronization strategies have been developed for chaotic systems. The sliding mode control method was applied to chaos control[18,19] and chaos synchronization[20]. In [21], the feedback control method and the adaptive control method were used to realize chaos control for the energy resource chaotic system. The chaos control problems were investigated for Lorenz system, Chen system and Lü system based on backstepping design method in [22]. By using adaptive control method, the problems of chaos control[23,24] and chaotic synchronization[25] were studied for chaotic systems. The neural adaptive control method was developed for a class of chaotic systems with uncertain dynamics, input and output constraints in [26]. In [27], on the basis of impulsive control method, the problems of the stabilization and synchronization were explored for Lorenz systems. The synchronization problem was resolved for a class of chaotic systems by using a fractional-order observer-based method and the synchronization was applied to secure communication in [28]. In [29], the synchronization was studied for fractional-order systems based on the output feedback sliding mode control method. A new synchronization strategy was presented for two fractional-order systems and the synchronization was applied in image encryption in [30]. The above literature works focused on chaos control and chaos synchronization in practical chaotic systems with equilibrium points. However, the control of chaotic systems without equilibrium has rarely been investigated[13]. Meanwhile, for most of the above mentioned works, fractional-order controllers have rarely been used to realize the chaos control of integer-order chaotic systems, although some important results on the fractional-order controllers have been proposed for various systems[31−33]. In [31], a well-known fractional-order controller was presented. In [32], the concept of a fractional-order $PI^{\lambda}D^{\mu}$ controller was proposed and the fractional-order controller included fractional order integrator and fractional-order differentiator. In [33], on the basis of the Lyapunov stability theory, a novel fractional-order controller was given, and fractional-order chaotic and hyperchaotic systems were controlled by the proposed fractional-order controller. The fractional-order controllers are effective to control systems, which have been proved in the mentioned works. Therefore, it is valuable to further explore the chaos control of integer-order chaotic systems without equilibrium by using fractional-order controllers.

Inspired by the above discussions, the objective of this paper

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is to design an efficient fractional-order controller and the stability is realized for the closed-loop system. A novel chaotic system without equilibrium is proposed based on the model of two-wheeled self-balancing robot. Meanwhile, the presented new system is used to verify the effectiveness of the proposed control scheme.

The organization of the paper is as follows. Section II details the problem formulation. A novel chaotic system is proposed and the chaotic system circuit is designed in Section III. Section IV presents the fractional-order controller based on the state-feedback method. The numerical simulation studies are presented to demonstrate the effectiveness of the developed control method in Section V, followed by some concluding remarks in Section VI.

II. Problem Statement and Preliminaries

In this paper, a novel chaotic system will be proposed by only considering the straight line position x_r and the pitch angle θ_p of the two-wheeled self-balancing robot of Googol Technology as shown in Fig. 1. A mathematical model related to the two-wheeled self-balancing robot of Googol Technology was established in [34]. The linear mathematical model for x_r and θ_p of the two-wheeled self-balancing robot of Googol Technology is described in the form

$$\begin{bmatrix} \dot{x}_r \\ \ddot{x}_r \\ \dot{\theta}_p \\ \ddot{\theta}_p \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -23.6701 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 124.5128 & 0 \end{bmatrix} \begin{bmatrix} x_r \\ \dot{x}_r \\ \theta_p \\ \dot{\theta}_p \end{bmatrix} + \begin{bmatrix} 0 \\ 4.5974 \\ 0 \\ -19.0414 \end{bmatrix} C_\theta \tag{1}$$

where C_θ denotes the pitch torque.

Fig. 1. Two-wheeled self-balancing robot of Googol Technology.

In order to transform the linear mathematical model into a chaotic system, we consider C_θ as a nonlinear term $\Phi(x_r, \dot{x}_r, \dot{\theta}_p, \ddot{\theta}_p)$, which will be given in next section.

Considering the nonlinear function $\Phi(x_r, \dot{x}_r, \dot{\theta}_p, \ddot{\theta}_p)$ and the control input u, (1) can be described as

$$\begin{bmatrix} \dot{x}_r \\ \ddot{x}_r \\ \dot{\theta}_p \\ \ddot{\theta}_p \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -23.6701 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 124.5128 & 0 \end{bmatrix} \begin{bmatrix} x_r \\ \dot{x}_r \\ \theta_p \\ \dot{\theta}_p \end{bmatrix} + \begin{bmatrix} 0 \\ 4.5974 \\ 0 \\ -19.0414 \end{bmatrix} \Phi(x_r, \dot{x}_r, \dot{\theta}_p, \ddot{\theta}_p) + u \tag{2}$$

where the control input $u = [u_1, u_2, u_3, u_4]^{\mathrm{T}}$.

This paper aims at constructing a novel chaotic system without equilibrium and developing a fractional-order control scheme, so that the stabilization of the whole closed-loop system is realized based on the designed control strategy. Under designed fractional-order controller, the state variables of the closed-loop system will be asymptotically stable. To develop the fractional-order control scheme, we firstly introduce the following definitions and lemmas:

Definition 1[35]**.** The Caputo fractional derivative operator, which is one of the most widely used fractional derivative operators, is defined for the function $f(t)$ as follows :

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau, \tag{3}$$

where α is the fractional order and $m-1 < \alpha < m$, $m = [\alpha]+1$, $[\alpha]$ denotes the integer part of α, and $\Gamma(\cdot)$ is gamma function, which is defined as $\Gamma(m-\alpha) = \int_0^\infty t^{m-\alpha-1} e^{-t} \mathrm{d}t$. The main advantage of (3) is that Caputo derivative of a constant is equal to zero. Particularly, when $0 < \alpha \leq 1$, we have $\mathcal{L}\{D^\alpha f(t)\} = s^\alpha F(s) - s^{\alpha-1} f(0)$. The Laplace transform of fractional integral at $t_0 = 0$ has the following form:

$$\mathcal{L}\left\{D^{-\alpha} f(t)\right\} = s^{-\alpha} \mathcal{L}\left\{f(t)\right\} = s^{-\alpha} F(s), \ \ (\alpha > 0), \tag{4}$$

where t and s are the variables in the time domain and Laplace domain, respectively. $F(s) = \mathcal{L}(f(t))$ and $\mathcal{L}(\cdot)$ stands for the Laplace transform.

In this paper, the fractional-order controller will be described by using Caputo definition with lower limit of integral $t_0 = 0$ and the order $1 < \alpha < 2$. Furthermore, there have been some important control schemes proposed for fractional-order systems by using different fractional calculus. In [36-38], Mittag-Leffler stability theorems have been proposed for fractional-order systems. The stability theorem was developed for fractional differential system with Riemann-Liouville derivative in [39-41].

Definition 2[42]**.** The Mittag-Leffler function with two parameters is defined as

$$E_{\alpha_1,\beta_1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha_1 + \beta_1)}, \tag{5}$$

where $\alpha_1 > 0$, $\beta_1 > 0$, z denotes the set of complex numbers. When $\beta_1 = 1$, the Mittag-Leffler function can be written as

$$E_{\alpha_1,1}(z) = E_{\alpha_1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha_1 + 1)}, \tag{6}$$

The Laplace transform of Mittag-Leffler function is given by

$$\begin{cases} \mathcal{L}\left\{t^{\beta_1-1}E_{\alpha_1,\beta_1}(-\lambda t^{\alpha_1})\right\} = \frac{s^{\alpha_1-\beta_1}}{s^{\alpha_1}+\lambda}, \Re(s) > |\lambda|^{\frac{1}{\alpha_1}}, \\ \mathcal{L}\left\{E_{\alpha_1,1}(-\lambda t^{\alpha_1})\right\} = \frac{s^{\alpha_1}}{s(s^{\alpha_1}+\lambda)}. \end{cases} \tag{7}$$

where $\Re(s)$ stands for the real part of s and $\lambda \in R$.

Lemma 1[43]**.** For the the Mittag-Leffler function $E_{\alpha_3,\beta_3}(A_0 t^{\alpha_3})$, if $1 < \alpha_3 < 2$, then, for $\beta_3 = 1$, 2 or α_3, one has

$$\|E_{\alpha_3,\beta_3}(A_0 t^{\alpha_3})\| \leq \left\| e^{A_0 t^{\alpha_3}} \right\|, \quad t \geq 0. \tag{8}$$

Moreover, if A_0 is a stable matrix, we have

$$\left\| e^{A_0 t^{\alpha_3}} \right\| \leq M e^{-\eta t}, \quad t \geq 0, \tag{9}$$

where $M \geq 1$, $-\eta(\eta > 0)$ is the largest eigenvalue of the matrix A_0, $\|\cdot\|$ denotes any vector or induced matrix norm.

Lemma 2[44,45]**(Gronwall-Bellman lemma).** Assume that the function $h(t)$ satisfies

$$h(t) \leq \int_0^t p(\tau)h(\tau)\mathrm{d}\tau + b(t), \tag{10}$$

with $p(\tau)$ and $b(t)$ being known real functions. Then, we obtain

$$h(t) \leq \int_0^t p(\tau)h(\tau)e^{\int_\tau^t p(\upsilon)\mathrm{d}\upsilon}\mathrm{d}\tau + b(t). \tag{11}$$

If $b(t)$ is differentiable, we have

$$h(t) \leq b(0)e^{\int_0^t p(\tau)\mathrm{d}\tau} + \int_0^t \dot{b}(\tau)e^{\int_\tau^t p(\upsilon)\mathrm{d}\upsilon}\mathrm{d}\tau. \tag{12}$$

In particular, if $b(t)$ is a constant, one has

$$h(t) \leq b(0)e^{\int_0^t p(\tau)\mathrm{d}\tau}. \tag{13}$$

III. Design of Chaotic System and Circuit Implementation

In this section, a novel chaotic system without equilibrium is constructed based on the linear mathematical model (1) of the two-wheeled self-balancing robot. For this case, the proposed novel chaotic system can be regarded as an open-loop system of the system (2). Furthermore, the chaotic circuit is designed to show the physical realizability of the proposed chaotic system.

A. A Novel Chaotic System

From (2), the novel chaotic system is described as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -23.6701x_3 + 4.5974\Phi(x) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= 124.5128x_3 - 19.0414\Phi(x) \end{aligned} \tag{14}$$

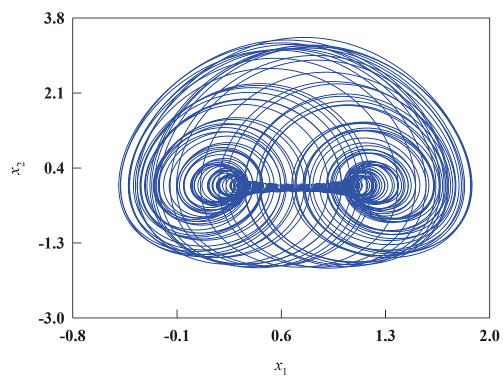
where $x = [x_1, x_2, x_3, x_4]^{\mathrm{T}}$ is the state vector of the nonlinear system with $x_1 = x_r$, $x_2 = \dot{x}_r$, $x_3 = \theta_p$ and $x_4 = \dot{\theta}_p$. The nonlinear function $\Phi(x)$ is given by

$$\Phi(x) = \kappa_1(x_2 + x_4 + x_1 x_3) + \kappa_2 \tag{15}$$

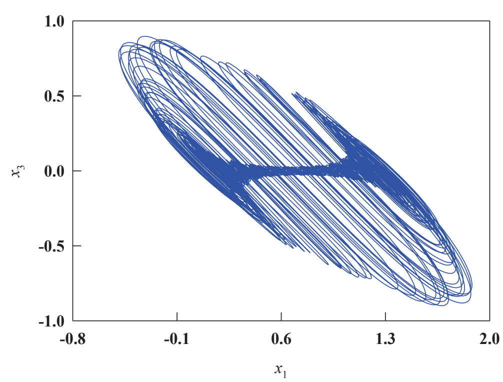
where κ_1 and κ_2 are constants. When $\kappa_1 = 10$ and $\kappa_2 = 0.5$, we obtain the Lyapunov exponents $\lambda_{L_1} = 0.0177$, $\lambda_{L_2} = 0$, $\lambda_{L_3} = -0.0148$ and $\lambda_{L_4} = -143.8384$ by using the initial conditions $x_{10} = x_{20} = x_{30} = x_{40} = 0.1$ based on the numerical method of [46]. Obviously, the system (14) is a chaotic system because $\lambda_{L_1} > 0$, $\lambda_{L_2} = 0$, $\lambda_{L_3} < 0$ and $\lambda_{L_4} < 0$. On the basis of the system (14) and the mentioned parameter values, some simulation results are further presented as shown in Fig. 2. In addition, to further reflect the properties of chaos, a Poincaré map is shown in Fig. 3.

In order to solve the equilibrium of system (14), we have $\dot{x}_1 = 0$, $\dot{x}_2 = 0$, $\dot{x}_3 = 0$ and $\dot{x}_4 = 0$, that is

$$\begin{aligned} x_2 &= 0 \\ -23.6701x_3 + 4.5974\Phi(x) &= 0 \\ x_4 &= 0 \\ 124.5128x_3 - 19.0414\Phi(x) &= 0 \end{aligned} \tag{16}$$



(a)



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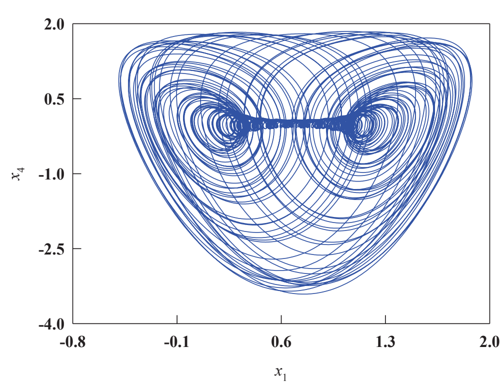

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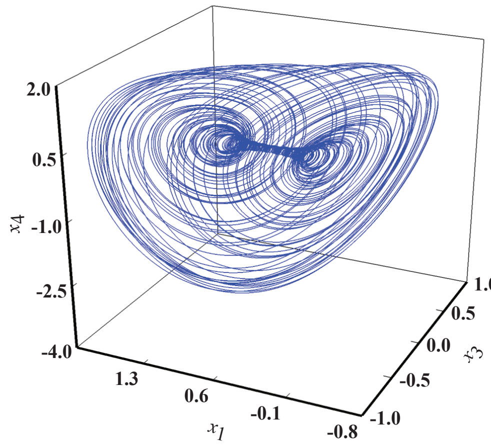


(d)

Fig. 2. Chaotic behaviors of the novel chaotic system (a) $x_1 - x_2$ plane, (b) $x_1 - x_3$ plane, (c) $x_1 - x_4$ plane, (c) $x_3 - x_1 - x_4$ space.

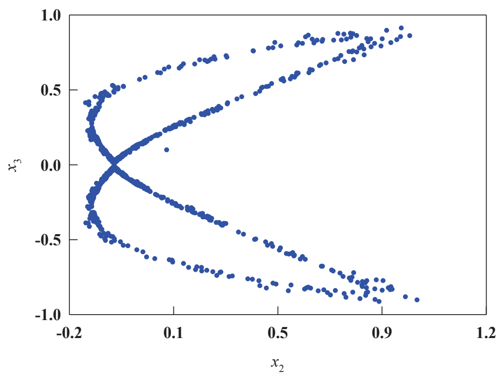


Fig. 3. Poincaré map in the $x_2 - x_3$ plane.

According to (16), we obtain that there is no equilibrium in system (14). Furthermore, we ensure that the system (14) is dissipative with the following exponential contraction rate:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = e^{-144.44t} \tag{17}$$

with

$$\begin{aligned} \nabla V &= \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{x}_3}{\partial x_3} + \frac{\partial \dot{x}_4}{\partial x_4} \\ &= -144.44 < 0 \end{aligned} \tag{18}$$

B. Circuit Implementation

To further illustrate the physical realizability of the proposed novel chaotic system (14), the system circuit is designed. By using the resistors, the capacitors and the operational amplifiers TL082, the designed circuit of the chaotic system is shown in Fig. 4. According to Fig. 4, the circuit system of the chaotic system is described as

$$\begin{aligned} \dot{x}_1 &= \frac{R_{12}}{C_1 R_{11} R_{13}} x_2 \\ \dot{x}_2 &= -\frac{1}{C_2 R_{28}} x_3 + \frac{R_{22}}{C_2 R_{23} R_{21}} x_2 + \frac{R_{26}}{C_2 R_{25} R_{27}} x_1 x_3 \\ &\quad + \frac{R_{210}}{C_2 R_{29} R_{211}} x_4 + V_1 \frac{1}{C_2 R_{24}} \\ \dot{x}_3 &= \frac{R_{32}}{C_3 R_{31} R_{33}} x_4 \\ \dot{x}_4 &= \frac{R_{42}}{C_4 R_{41} R_{44}} x_3 - \frac{1}{C_4 R_{46}} x_2 - \frac{1}{C_4 R_{45}} x_1 x_3 \\ &\quad - \frac{1}{C_4 R_{47}} x_4 - V_2 \frac{1}{C_4 R_{43}} \end{aligned} \tag{19}$$

By comparing (14) with (19), all resistance values R_{11}, R_{12}, R_{21}, R_{22}, R_{25}, R_{26}, R_{29}, R_{210}, R_{31}, R_{32}, R_{41} and R_{42} are $10\,K\Omega$, R_{13} and R_{33} are $1\,M\Omega$, R_{23}, R_{27} and R_{211} are $21.7514\,K\Omega$, R_{45}, R_{46} and R_{47} are $5.251715\,K\Omega$, R_{28} is $42.2474\,K\Omega$, R_{44} is $8.031303\,K\Omega$, R_{24} is $435.02849\,K\Omega$ and R_{43} is $105.0343\,K\Omega$. The voltage values $V_1 = 1\,V$ and $V_2 = -1\,V$. In order to speed up the circuit response time, we make a time scale transformation by multiplying a factor of 100 on the right hand side of (14), the capacitance values C_1, C_2, C_3 and C_4 are $10\,nF$. In Fig. 4, $U_{Ai}(i = 1, 2, \cdots, 10)$ are operational amplifiers, A is a unity gain multiplier.

From the designed circuit of chaotic system (14), the circuit experimental phase portraits are presented in Fig. 5. Comparing Fig. 2 and Fig. 5, we observe that there exists consistency between numerical simulations and circuit experimental simulations, the circuit simulation results prove the physical realizability of the proposed novel chaotic system (14).

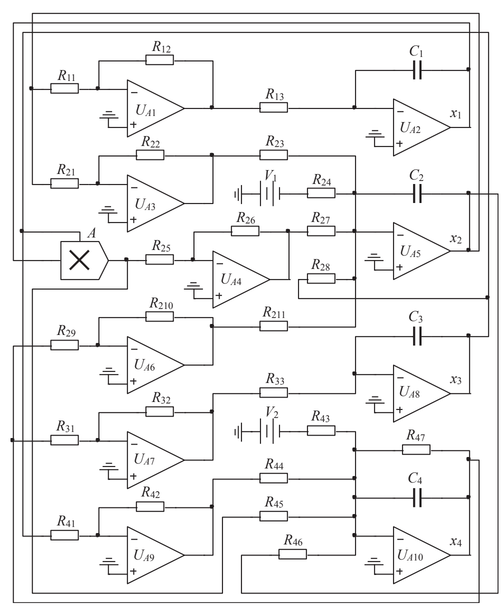
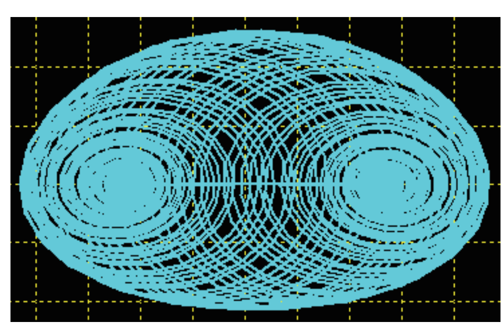
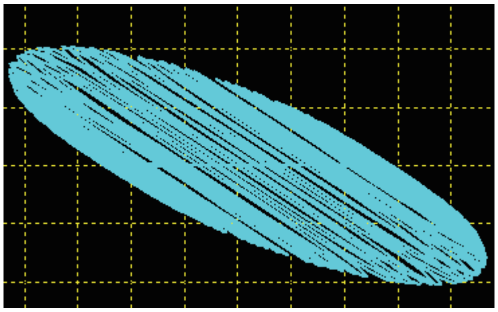


Fig. 3. Circuit of the novel chaotic system (14).



(a)



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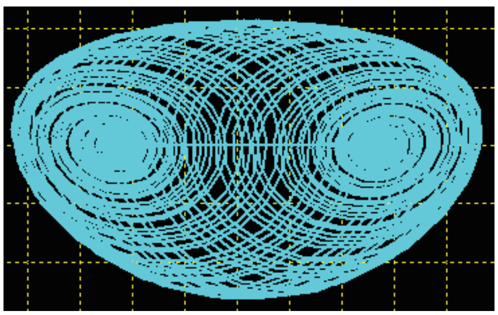

(c)

Fig. 5. Chaotic behaviors of the chaotic circuit (a) $x_1 - x_2$ plane, (b) $x_1 - x_3$ plane, (c) $x_1 - x_4$ plane.

IV. Design of Fractional-order Controller and Stability Analysis

In this section, the control scheme will be proposed for the whole closed-loop system including the constructed chaotic system (14) and the designed fractional-order controller. The goal is to guarantee the stabilization of the closed-loop system under the proposed fractional-order controller.

From (14), the chaotic system can be rewritten as

$$\dot{x} = Ax + q(x) + \bar{q} \tag{20}$$

where $x = [x_1, x_2, x_3, x_4]^{\mathrm{T}}$ is the state vector,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 45.974 & -23.6701 & 45.974 \\ 0 & 0 & 0 & 1 \\ 0 & -190.414 & 124.5128 & -190.414 \end{bmatrix}$$

$$q(x) = \begin{bmatrix} 0 \\ 45.974x_1x_3 \\ 0 \\ -190.414x_1x_3 \end{bmatrix}$$

$$\bar{q} = \begin{bmatrix} 0 \\ 2.2987 \\ 0 \\ -9.5207 \end{bmatrix}.$$

According to the chaotic system (14) and considering the control input u, the corresponding system has the following form:

$$\dot{x} = Ax + q(x) + \bar{q} + u \tag{21}$$

where $u = [u_1, u_2, u_3, u_4]^{\mathrm{T}}$ is the designed fractional-order control input.

Based on the state-feedback control method, the controller u is defined as

$$u = -Ax - \bar{q} + KD^{1-\alpha}x. \tag{22}$$

where $K = \mathrm{diag}(k_1, k_2, \cdots, k_n)$ is a design control gain matrix and the fractional order satisfies $1 < \alpha < 2$.

According to (21) and (22), one has

$$\dot{x} = q(x) + KD^{1-\alpha}x. \tag{23}$$

To render the stabilization of the system (21) under the proposed controller (22), the following assumption is required:

Assumption 1. Nonlinear function $q(x)$ satisfies $q(0) = 0$ and $\lim_{\|x\|\to 0}\|q(x)\|/\|x\| = 0$.

The fractional-order controller based control scheme for the closed-loop system (23) can be summarized in the following theorem.

Theorem 1. For the closed-loop system (23), the fractional-order controller is designed based on (22). Then, the state variables of the closed-loop system (23) are asymptotically stabilized to zero when the zero is a stable equilibrium point of the closed-loop system (23), under the conditions of $\lim_{\|x\|\to 0}\|q(x)\|/\|x\| = 0$, the fractional order α: $1 < \alpha < 2$, and the design matrix K satisfies $\eta = -\max\{\mathrm{Re}\lambda(K)\} > 1$, where $\lambda(K)$ denotes the eigenvalues of K and $M \geq 1$.

Proof. By taking the Laplace transform on system (23), we have

$$sX(s) - x(0) = \mathcal{L}(q(x(t))) + Ks^{1-\alpha}X(s), \tag{24}$$

where $X(s)$ is the Laplace transform of $x(t)$, $x(0)$ is the initial condition of (17) and $1 < \alpha < 2$.

Let us multiply both sides of (24) by s^{α}, it yields

$$s^{\alpha+1}X(s) - s^{\alpha}x(0) = s^{\alpha}\mathcal{L}(q(x(t))) + KsX(s). \tag{25}$$

From (25), one has

$$X(s) = s^{\alpha-1}(Is^{\alpha} - K)^{-1}(x(0) + \mathcal{L}(q(x(t)))), \tag{26}$$

where I denotes the 4×4 identity matrix.

Taking the Laplace inverse transform on (26), one obtains

$$\begin{aligned} x(t) \quad &= \quad E_{\alpha,1}(Kt^{\alpha})x(0) \\ &\quad + \int_0^t E_{\alpha,1}(K(t-\varphi)^{\alpha})q(x(\varphi))\mathrm{d}\varphi. \end{aligned} \tag{27}$$

On the basis of Lemma 1, since K is a stable matrix, $-\eta = \max(\mathrm{Re}\lambda(K))(\eta > 0)$, $M \geq 1$ and $1 < \alpha < 2$, (27) can be written as

$$\|x(t)\| \leq Me^{-\eta t}\|x(0)\| + \int_0^t Me^{-\eta(t-\varphi)}\|q(x(\varphi))\|\,\mathrm{d}\varphi. \tag{28}$$

Multiplying by $e^{\eta t}$ on both sides of (28), it yields

$$e^{\eta t}\|x(t)\| \leq M\|x(0)\| + \int_0^t Me^{\eta\varphi}\|q(x(\varphi))\|\,\mathrm{d}\varphi. \tag{29}$$

According to Assumption 1 and the properties of $\lim_{\|x\|\to 0}\|q(x)\|/\|x\| = 0$[43,47], there exists a constant $\delta > 0$, such that

$$\|q(x)\| \leq \frac{1}{M}\|x\| \; as \; \|x\| < \delta. \tag{30}$$

Substituting (30) into (29), one has

$$e^{\eta t}\left\|x(t)\right\| \leq M\left\|x(0)\right\| + \int_0^t e^{\eta\varphi}\left\|x(\varphi)\right\| \mathrm{d}\varphi. \tag{31}$$

Based on Lemma 2, $b(t) = M\left\|x(0)\right\|$, $p(\varphi) = 1$ and $h(t) = e^{\eta t}\left\|x(t)\right\|$, we have

$$e^{\eta t}\left\|x(t)\right\| \leq M\left\|x(0)\right\| e^{t}. \tag{32}$$

Inequality (32) is equivalent to

$$\left\|x(t)\right\| \leq \frac{M\left\|x(0)\right\|}{e^{(\eta-1)t}}. \tag{33}$$

When $\eta = -\max\left\{\mathrm{Re}\lambda(K)\right\} > 1$, $t \to \infty$, $\left\|x(t)\right\|$ asymptotically tends to zero, which implies the closed-loop system (23) is asymptotically stable if zero is a stable equilibrium point. This concludes the proof.

V. Numerical Simulation

In this section, in order to illustrate and verify the effectiveness of the proposed control scheme, the closed-loop system (23) is analyzed. Furthermore, we use the proposed control scheme to stabilize the chaotic systems with equilibrium such as Chen system[48], Genesio's system[49], and hyperchaotic Lorenz system[50].

A. Novel chaotic system

Combining the novel chaotic system (14) and the designed controller (22), we have

$$\begin{aligned} \dot{x}_1 &= k_1 D^{1-\alpha} x_1 \\ \dot{x}_2 &= 45.974 x_1 x_3 + k_2 D^{1-\alpha} x_2 \\ \dot{x}_3 &= k_3 D^{1-\alpha} x_3 \\ \dot{x}_4 &= -190.414 x_1 x_3 + k_4 D^{1-\alpha} x_4 \end{aligned} \tag{34}$$

The equilibrium of system (34) is obtained by solving $\dot{x}_1 = 0$, $\dot{x}_2 = 0$, $\dot{x}_3 = 0$ and $\dot{x}_4 = 0$, that is

$$\begin{aligned} k_1 D^{1-\alpha} x_1 &= 0 \\ 45.974 x_1 x_3 + k_2 D^{1-\alpha} x_2 &= 0 \\ k_3 D^{1-\alpha} x_3 &= 0 \\ -190.414 x_1 x_3 + k_4 D^{1-\alpha} x_4 &= 0 \end{aligned} \tag{35}$$

According to (35), we obtain that $O = (0, 0, 0, 0)$ is the equilibrium of the system (34). Furthermore, when the design parameters k_1, k_2, k_3 and k_4 satisfy $k_1 < 0$, $k_2 < 0$, $k_3 < 0$ and $k_4 < 0$, we can guarantee that the equilibrium $O = (0, 0, 0, 0)$ is a stable equilibrium based on the stability analysis method of the equilibrium[51].

From (34), we have

$$\begin{aligned} \lim_{\|x\|\to 0} \frac{\|q(x)\|}{\|x\|} &= \lim_{\|x\|\to 0} \frac{\sqrt{38371 x_1^2 x_3^2}}{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}} \\ &\leq \lim_{\|x\|\to 0} \frac{\sqrt{38371 x_1^2 x_3^2}}{\sqrt{x_3^2}} \\ &= \lim_{\|x\|\to 0} 195.8854\left|x_1\right| = 0 \end{aligned} \tag{36}$$

which implies that $q(x)$ satisfies Assumption 1. On the basis of Theorem 1 and pole placement technique, the feedback control gain matrix and the order α are chosen as

$$K = \mathrm{diag}(-10, -10, -10, -10), \quad \alpha = 1.6 \tag{37}$$

From the above discussion, we have $\left\|e^{Kt^{\alpha}}\right\| \leq e^{-10t}$, $M = 1$ and $\eta = -\max\left\{\mathrm{Re}\lambda(K)\right\} = 10 > 1$, which satisfy Theorem 1. The simulation results are shown in Fig. 6 and Fig. 7. According to the numerical simulation results, the closed-loop system (34) is asymptotically stable, which implies that the proposed control scheme works effectively.

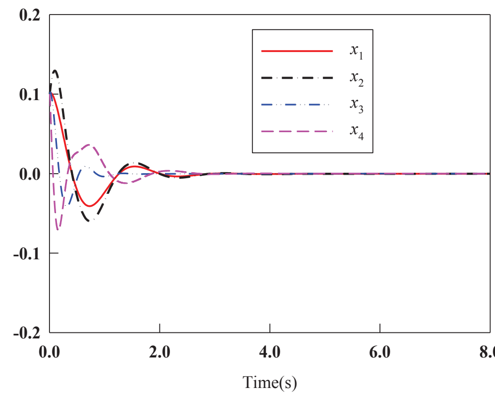

Fig. 6. Numerical simulation results of the system (34).

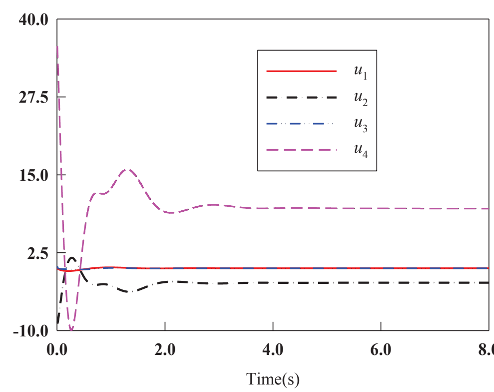

Fig. 7. Control inputs.

B. Chaotic systems with equilibrium

In order to further illustrate the effectiveness of the developed control scheme in this paper, we use the proposed control scheme (22) to control Chen system[48], Genesio's system[49], and hyperchaotic Lorenz system[50]. We firstly analyze the following dynamical model of Chen system[48]:

$$\begin{aligned} \dot{x}_1 &= 35(x_2 - x_1) \\ \dot{x}_2 &= -7x_1 - x_1 x_3 + 28 x_2 \\ \dot{x}_3 &= x_1 x_2 - 3x_3 \end{aligned} \tag{38}$$

From (22), the control input u is designed for the Chen system as follows:

$$\begin{aligned} u_1 &= -35(x_2 - x_1) + k_1 D^{1-\alpha} x_1 \\ u_2 &= 7x_1 - 28x_2 + k_2 D^{1-\alpha} x_2 \\ u_3 &= 3x_3 + k_3 D^{1-\alpha} x_3 \end{aligned} \tag{39}$$

Invoking (38), we have

$$\lim_{\|x\|\to 0} \frac{\sqrt{x_1^2 x_2^2 + x_1^2 x_3^2}}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \le \lim_{\|x\|\to 0} |x_1| = 0 \tag{40}$$

where $x = [x_1, x_2, x_3]^{\mathrm{T}}$.

According to (40), the nonlinear function in (38) can satisfy the Assumption 1. Therefore, the Chen system (38) can be stabilized to zero by choosing appropriate parameters k_1, k_2 and k_3.

The Genesio's system is written by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -6x_1 - 2.92x_2 - 1.2x_3 + x_1^2 \end{aligned} \tag{41}$$

To control the Genesio's system (41), the control input u can be designed based on (22) as

$$\begin{aligned} u_1 &= -x_2 + k_1 D^{1-\alpha} x_1 \\ u_2 &= -x_3 + k_2 D^{1-\alpha} x_2 \\ u_3 &= 6x_1 + 2.92x_2 + 1.2x_3 + k_3 D^{1-\alpha} x_3 \end{aligned} \tag{42}$$

From (41), we obtain

$$\lim_{\|x\|\to 0} \frac{\sqrt{x_1^4}}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \le \lim_{\|x\|\to 0} |x_1| = 0 \tag{43}$$

where $x = [x_1, x_2, x_3]^{\mathrm{T}}$.

The nonlinear function in (41) can satisfy the Assumption 1 based on (43). Thus, the Genesio's system (41) can be stabilized to zero under the appropriate parameters k_1, k_2 and k_3.

The hyperchaotic Lorenz system is given as follows:

$$\begin{aligned} \dot{x}_1 &= 10(x_2 - x_1) \\ \dot{x}_2 &= 28x_1 - x_1 x_3 - x_2 \\ \dot{x}_3 &= x_1 x_2 - \frac{8}{3} x_3 \\ \dot{x}_4 &= -x_1 x_3 + 1.2x_4 \end{aligned} \tag{44}$$

Combining the hyperchaotic Lorenz system (44) and the control scheme (22), the control input u is written as

$$\begin{aligned} u_1 &= -10(x_2 - x_1) + k_1 D^{1-\alpha} x_1 \\ u_2 &= -28x_1 + x_2 + k_2 D^{1-\alpha} x_2 \\ u_3 &= \frac{8}{3} x_3 + k_3 D^{1-\alpha} x_3 \\ u_4 &= -1.2x_4 + k_4 D^{1-\alpha} x_4 \end{aligned} \tag{45}$$

According to (44), we have

$$\lim_{\|x\|\to 0} \frac{\sqrt{2x_1^2 x_3^2 + x_1^2 x_2^2}}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \le \lim_{\|x\|\to 0} \sqrt{2x_2^2 + x_3^2} = 0 \tag{46}$$

where $x = [x_1, x_2, x_3, x_4]^{\mathrm{T}}$.

On the basis of (46), the Assumption 1 is satisfied for the nonlinear function in (44). By designing appropriate parameters k_1, k_2, k_3 and k_4, the stabilization of the hyperchaotic Lorenz system (44) can be realized.

According to the above discussion and analysis, we obtain that the Chen system (38), the Genesio's system (41) and the hyperchaotic Lorenz system (44) can be controlled by using the designed control scheme in this paper. For the numerical simulation of the Chen system (38), we choose the control parameters $k_1 = -10$, $k_2 = -10$ and $k_3 = -10$, the initial conditions $x_0 = (3, 2, 3)^{\mathrm{T}}$ and the fractional order $\alpha = 1.6$. For the numerical simulation of the Genesio's system (41), we set the control parameters as $k_1 = -10$, $k_2 = -10$ and $k_3 = -10$, the initial conditions as $x_0 = (-1, -1, 0)^{\mathrm{T}}$ and the fractional order $\alpha = 1.6$. The control parameters are designed as $k_1 = -10$, $k_2 = -10$, $k_3 = -10$ and $k_4 = -10$, the initial conditions are assumed as $x_0 = (0.1, 0.1, 0.1, 0.1)^{\mathrm{T}}$ and the fractional order is chosen as $\alpha = 1.6$ in the numerical simulation of the hyperchaotic Lorenz system (44).

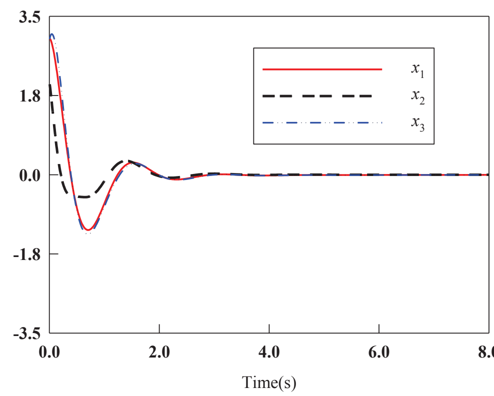

Fig. 8. Stabilization of Chen system (38).

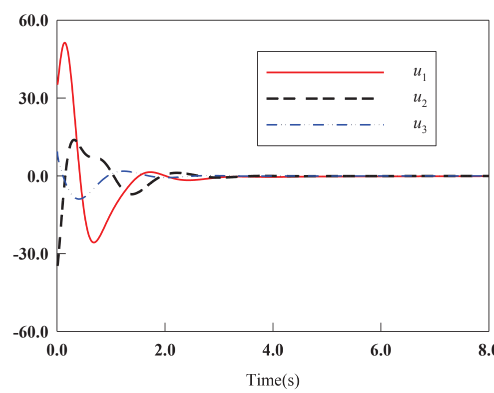

Fig. 9. Control inputs of Chen system (38).

On the basis of the above given simulation conditions, the numerical results are presented in Fig. 8-Fig. 13 for the Chen system (38), the Genesio's system (41), and the hyperchaotic Lorenz system (44). The control result of the Chen system (38) is shown in Fig. 8. It is illustrated that good control performance is obtained under the designed controller (39). Fig. 9 presents the control input (39). The numerical results of the Genesio's system (41) are given in Fig. 10 and Fig. 11. Fig. 10 and Fig. 11 show that the controller (42) can stabilize the Genesio's system (41) well. Finally, Fig. 12 and Fig. 13 show that the fractional-order controller (45) can control all state variables of the hyperchaotic Lorenz system (44) to the origin point. Therefore, all the simulation results show that the fractional-order controller also can control the chaotic and hyperchaotic systems with equilibrium.

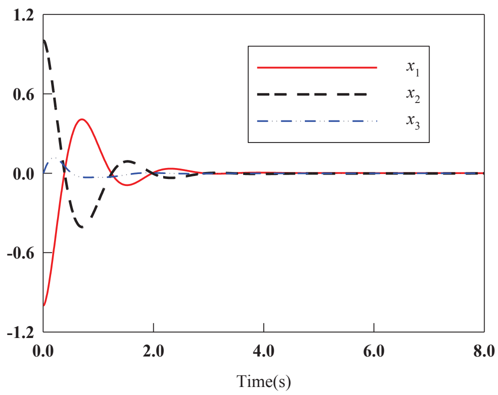

Fig. 10. Stabilization of Genesio's system (41).

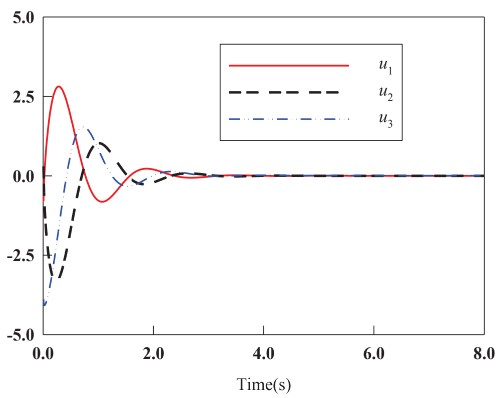

Fig. 11. Control inputs for Genesio's system (41).

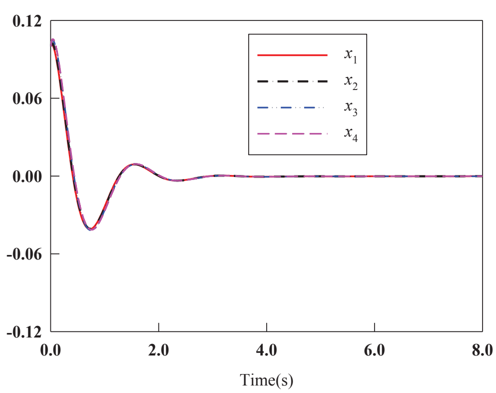

Fig. 12. Stabilization of hyperchaotic Lorenz system (44).

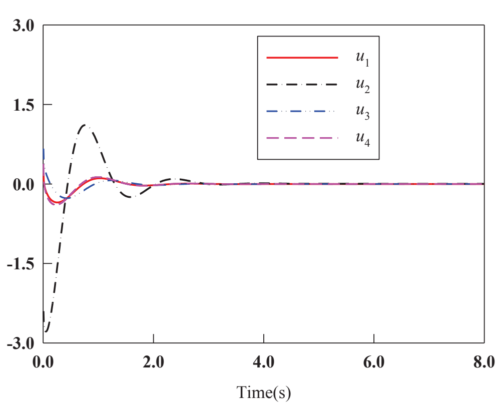

Fig. 13. Control inputs for hyperchaotic Lorenz system (44).

VI. CONCLUSION

In this paper, a novel chaotic system without equilibrium has been proposed. The Lyapunov exponents and Poincaré map of the proposed chaotic system have been given. Meanwhile, the dissipativeness of the new chaotic system has been illustrated. The chaotic circuit has been designed to demonstrate the physical realizability of the novel chaotic system. In addition, on the basis of the Gronwall inequality, the Laplace transform, the Mittag-Leffler function and the state-feedback method, a stability theorem for a class of closed-loop systems has been given. The designed controller has been developed to realize the stabilization of the closed-loop system. Furthermore, the proposed control scheme has been developed to control the chaotic and hyperchaotic systems with equilibrium, i.e. Chen system, Genesio's system and hyperchaotic Lorenz system. Finally, the numerical simulation results further illustrate the effectiveness of the developed control scheme.

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