# Modified Grey Model Predictor Design Using Optimal Fractional-order Accumulation Calculus

Yang Yang, and Dingyü Xue

*Abstract*—**The major advantage of grey system theory is that both incomplete information and unclear problems can be processed precisely. Considering that the modeling of grey model (GM) depends on the preprocessing of the original data, the fractional-order accumulation calculus could be used to do preprocessing. In this paper, the residual sequence represented by Fourier series is used to ameliorate performance of the fractional-order accumulation $\mathrm{GM}(1,1)$ and improve the accuracy of predictor. The state space model of optimally modified $\mathrm{GM}(1,1)$ predictor is given and genetic algorithm (GA) is used to find the smallest relative error during the modeling step. Furthermore, the fractional form of continuous $\mathrm{GM}(1,1)$ is given to enlarge the content of prediction model. The simulation results illustrated that the fractional-order calculus could be used to depict the GM precisely with more degrees of freedom. Meanwhile, the ranges of the parameters and model application could be enlarged with better performance. The method of modified GM predictor using optimal fractional-order accumulation calculus is expected to be widely used in data processing, model theory, prediction control and related fields.**

*Index Terms*—**Fractional-order accumulation, grey model (GM), genetic algorithm (GA), fourier series.**

## I. Introduction

GREY system theory was firstly proposed in 1982 and GM was built for prediction or decision-making with unclear and incomplete information [1]. Compared with the conventional statistical models, only small samples of data is required to estimate the behavior of unknown systems in the GM using a special differential equation according to the output data predicted [1]−[4]. $\mathrm{GM}(1,1)$ is the basic form and most commonly used due to its computational efficiency and prediction accuracy [3]−[5]. Furthermore, extraordinary differential equations (for example Non-linear, Delayed and Fractional-order) are also popularly used in mathematical modeling of many engineering and scientific problems.

Although, fractional-order differential equations (FODEs) have been efficient tools for model simulation and theoretical analysis, it is difficult to get the solution due to the existence of fractional derivatives, especially with the initial value problem [6]−[9]. Several Matlab programs and interfaces have been given for fractional-order systems. For example, Simulink model method for fractional-order nonlinear system is proposed in ref [10] for initial value problem. A Ninteger toolbox is developed for fractional-order controllers [11] and FOMCON is a modeling and control toolbox for fractional-order system [12]. The state space representation [13], [14], robust stability research and system analysis are also hot topics. In ref [13], two methods for the state space representation are presented based on the differentiation and integration operator approximation respectively. The robust stability of Fractional-order Linear Time Invariant (FO-LTI) system with interval uncertainties has been investigated in ref [14], [15]. Furthermore, parameter and differentiation order estimation in fractional models are discussed in ref [16].

Based on the fractional-order accumulation, many researches have been carried out to study the model performance in the GMs, for example, the model properties, perturbation problems and stability analysis of the fractional-order accumulation calculus by $\alpha = p/q$ [17]−[22]. Furthermore, several new GMs are proposed based on the fractional-order accumulation, for example, non-homogenous discrete grey model (NDGM) with fractional-order accumulation is put forward in ref [19], the fractional-order accumulating generation operator is applied in the $\mathrm{GM}(2,1)$ in ref [20], the grey discrete power $\mathrm{GM}(1,1)$ model is constructed by the fractional-order accumulation in ref [21], the fractional-order accumulation time-lag model $\mathrm{GM}(1,\mathrm{N},\tau)$ is proposed in [22]. Apart from this, the residual information are also used in some modified GMs. For example, Fourier residual, Markov Chain model and Artificial Neural Network have been used to correct the periodicity and randomness of residuals and improve the model performance [23]−[25].

Compared with the work being published, the GM using fractional-order accumulation, the modified optimal model and the model state space descriptions are all practical questions. In order to settle the problems above, the works described in this paper include the discrete GM $(1,1)$ using fractional-order accumulation and the fractional form of continuous GM $(1,1)$ by fractional calculus are studied; the modified optimal model is obtained based on GA and Fourier series; the state space models for the predictor are also given.

The rest of this paper is organized as follows. In Section II, the basic knowledge of fractional accumulation calculus is introduced. In Section III, the form and model by fractional-order accumulation calculus in GM $(1,1)$ is analyzed. In Sec-

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tion IV, the state space models of modified optimal fractional accumulation GM $(1,1)$ predictor are studied. In Section V, the fractional form of continuous GM $(1,1)$ is given. In Section VI, the properties of fractional-order accumulation calculus are showed by a simple sequence and one case of modified optimal fractional-order accumulation GM $(1,1)$ predictor is discussed and tested. The conclusion part is given finally.

## II. Fractional Accumulation Calculus

### A. Fractional Calculus

Fractional calculus is a generalization of differentiation and integration to non-integer-order fundamental operator. The continuous differential-integral operator is

$$ {}_aD_t^{\gamma} = \frac{\mathrm{d}^{\gamma}}{\mathrm{d}t^{\gamma}}, \qquad \gamma > 0 \tag{1a}$$

$$ {}_aD_t^{\gamma} = 1, \qquad \gamma = 0 \tag{1b}$$

$$ {}_aD_t^{\gamma} = \int_a^t (\mathrm{d}\tau)^{-\gamma}, \qquad \gamma < 0 \tag{1c}$$

where $\gamma$ is a complex number, $a$ is a real number related to initial value. $\Gamma(n)$ is Euler's Gamma function and

$$\Gamma(n) = \int_o^{\infty} t^{n-1}e^{-t}\mathrm{d}t \text{ for } \mathrm{Re}(n) > 0 \tag{2}$$

It is also called Euler's integral of the second kind. It is clear that the restriction $\mathrm{Re}(n) > 0$ assures the convergence of the integral. The Grünwald–Letnikov (GL), the Riemann–Liouville (RL) or Caputo derivative are the three most frequently used definitions for the general fractional differintegral, and the GL definition [26]−[28] is given as

$$ {}_aD_t^{\gamma} f(\mathrm{t}) = \frac{\mathrm{d}^{\gamma} f(t)}{\mathrm{d}t^{\gamma}} = \lim_{h\to 0}\frac{1}{h^{\gamma}}\sum_{j=0}^{(t-a)/h}(-1)^j\binom{\gamma}{j}f(t-jh) \tag{3}$$

where the binomial coefficients are defined as

$$\binom{\gamma}{j} = \frac{\gamma!}{j!\,(\gamma-j)!} \tag{4}$$

For arbitrary non-integer and even complex $\gamma \neq -1, -2, \cdots$ and $j$:

$$\begin{aligned}\binom{\gamma}{j} &= \frac{\Gamma(1+\gamma)}{\Gamma(1+j)\Gamma(1+\gamma-j)} \\ &= \frac{\sin[(j-\gamma)\pi]}{\pi}\frac{\Gamma(1+\gamma)\Gamma(j-\gamma)}{\Gamma(1+j)}\end{aligned} \tag{5}$$

For an integer $j = n$ and non-integer $\gamma$

$$\begin{aligned}\binom{\gamma}{n} &= \frac{\gamma(\gamma-1)\cdots(\gamma-n+1)}{n!} \\ &= \frac{(-1)^{n-1}\mu\Gamma(n-\gamma)}{n!\Gamma(1-\mu)} = O(n^{-\mu-1}), \quad n\to\infty\end{aligned} \tag{6}$$

### B. Fractional Accumulation Calculus

In the accumulation theory, for $\alpha$ is arbitrary positive integer number, the definition of integral-order accumulation for sequence $x(j)$ $j = 1, 2, \cdots, m$ is given as in (7) [29].

$$\begin{aligned}
\sum_{j=1}^{m}{}^{(1)}x(j) &= \sum_{j=1}^{m} C_{m-j}^{m-j}x(j) = \sum_{j=1}^{m}x(j) \quad \alpha = 1\\
\sum_{j=1}^{m}{}^{(2)}x(j) &= \sum_{j=1}^{m}\sum_{i=1}^{j}{}^{(1)}x(j) = \sum_{j=1}^{m}C_{m-j+1}^{m-j}x(j)\\
&= \sum_{j=1}^{m}C_{m-j+1}^{1}x(j) \quad \alpha = 2\\
&\vdots\\
\sum_{j=1}^{m}{}^{(k)}x(j) &= \sum_{j=1}^{m}\sum_{i=1}^{j}{}^{(k-1)}x(j)\\
&= \frac{1}{(k-1)!}\sum_{j=1}^{m}(m-j+1)(m-j+2)\\
&\quad\cdots[m-j+(k-1)]x(j)\\
&= \sum_{j=1}^{m}C_{m-j+k-1}^{m-j}x(j)\\
&= \frac{(m-j+k-1)(m-j+k-2)\cdots(k+1)k}{(m-j)!}\\
&\quad\cdot x(j) \quad \alpha = k
\end{aligned} \tag{7}$$

where $\alpha = 1, 2, \cdots, k$ are the integer-orders. For $\alpha = \frac{p}{q}$, the accumulation for sequence $x(j)$ $j = 1, 2, \cdots, m$ has been given in ref [17] and the definition is

$$\sum_{j=1}^{m}{}^{(\frac{p}{q})}x(j) = \sum_{j=1}^{m}C_{m-j+\frac{p}{q}-1}^{m-j}x(j) \tag{8}$$

where

$$C_{\frac{p}{q}-1}^{0} = 1,$$

$$C_{m-j+\frac{p}{q}-1}^{m-j} = \frac{(m-j+\frac{p}{q}-1)(m-j+\frac{p}{q}-2)\cdots(\frac{p}{q}+1)\frac{p}{q}}{(m-j)!}.$$

Based on the fractional-order and accumulation theory [9], [17], [29], the definition of fractional-order accumulation is given by Gamma function in this paper. For sequence $x(j)$ $j = 1, 2, \cdots, m$, the fractional-order accumulation is defined as in (9) with the order $k > 0$.

$$\sum_{j=1}^{m}{}^{(k)}x(j) = \sum_{j=1}^{m}\frac{\Gamma(m-j+k)}{\Gamma(m-j+1)\Gamma(k)}x(j) \tag{9}$$

and

$$\sum_{j=1}^{m}{}^{(k)} = \frac{\Gamma(m-j+k)}{\Gamma(m-j+1)\Gamma(k)} = \tag{10}$$

$$\begin{cases} C_{k-1}^{0} = 1 & m = j\\ C_{m-j+k-1}^{m-j} = \frac{(m-j+k-1)(m-j+k-2)\cdots(k+1)k}{(m-j)!} & m \neq j\end{cases}$$

The proof can be easily obtained using the properties of Gamma function. The expression by (9) is generalization of integer or $\alpha = \frac{p}{q}$. The parameter $w_j$ is defined as the weighting factor of accumulation as in (11).

$$w_j = \frac{\Gamma(m-j+k)}{\Gamma(m-j+1)\Gamma(k)} \quad j = 1, 2, \cdots, m \tag{11}$$

Using the fractional-order calculus, the weighted form of the overall data information is considered. Therefore, the fractional-order accumulation can be used to do data preprocessing and mine the information from the data. The estimated data by the model can be obtained from the following equation.

$$\hat{x}^{(0)}(m) = x^{(1)}(m) - \sum_{j=1}^{m-1} \frac{\Gamma(m-j+k)}{\Gamma(m-j+1)\Gamma(k)} \hat{x}^{(0)}(j) \tag{12}$$

## III. Fractional-order Accumulation in GM $(1,1)$

Suppose $x^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(m)\right)$ is the sequence of raw data with non-nonnegative values usually. Denote its fractional-order accumulation generated sequence by $x^{(1)} = \left(x^{(1)}(1), x^{(1)}(2), \cdots, x^{(1)}(m)\right)$.

The image form for continuous data of GM $(1,1)$ is [30]–[32].

$$\frac{\mathrm{d}x^{(1)}}{\mathrm{d}t} + ax^{(1)} = b \tag{13}$$

where $a$ and $b$ are referred as the development coefficient and grey action quantity, respectively. Using the difference instead of differential form, $\Delta t = (t+1) - t = 1$, as in (13), it can be rewritten as $x^{(1)}(i+1) - x^{(1)}(i) + ax^{(1)}(i+1) = b$, which is the basic form of the fractional-order accumulation GM $(1,1)$. The matrix form can be given as

$$\begin{bmatrix} x^{(1)}(2) - x^{(1)}(1) \\ x^{(1)}(3) - x^{(1)}(2) \\ \vdots \\ x^{(1)}(m) - x^{(1)}(m-1) \end{bmatrix} = \begin{bmatrix} -x^{(1)}(2) & 1 \\ -x^{(1)}(3) & 1 \\ \vdots & 1 \\ -x^{(1)}(m) & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{14}$$

Suppose

$$xx(j) = \frac{\Gamma(m-j+k)}{\Gamma(m-j+1)\Gamma(k)} x(j) = w_j x(j) \quad j = 1, 2, \cdots, m.$$

By using the least squares to estimate the model as in (14), it satisfies that

$$[a \quad b]^T = B^{-1} Y \tag{15}$$

where

$$B = \begin{bmatrix} -x^{(1)}(2) & 1 \\ -x^{(1)}(3) & 1 \\ \vdots & \vdots \\ -x^{(1)}(m) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\sum_{j=1}^{2} \frac{\Gamma(m-j+k)}{\Gamma(m-j+1)\Gamma(k)} x(j) & 1 \\ -\sum_{j=1}^{3} \frac{\Gamma(m-j+k)}{\Gamma(m-j+1)\Gamma(k)} x(j) & 1 \\ \vdots & \vdots \\ -\sum_{j=1}^{m} \frac{\Gamma(m-j+k)}{\Gamma(m-j+1)\Gamma(k)} x(j) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{(m-1)\times m} \begin{bmatrix} -xx(1) & 1 \\ -xx(2) & 0 \\ \vdots & \vdots \\ -xx(m) & 0 \end{bmatrix}_{m\times 2}$$

$$Y = [x^{(1)}(2) - x^{(1)}(1), x^{(1)}(3) - x^{(1)}(2), \cdots, \\ x^{(1)}(m) - x^{(1)}(m-1)]^T.$$

If $k = 1$, the model by (15) becomes the traditional one-order GM $(1,1)$.

Denote the fractional-order accumulation generated sequence by $x^{(1)}$ and $z^{(1)}$, $z^{(1)}$ is the average value of the adjacent neighbors of $x^{(1)}(k)$ and it can be expressed as

$$z^{(1)}(i) = \frac{x^{(1)}(i) + x^{(1)}(i+1)}{2}, \quad i = 1, 2, \cdots, m-1.$$

The matrix form is

$$\begin{bmatrix} x^{(1)}(2) - x^{(1)}(1) \\ x^{(1)}(3) - x^{(1)}(2) \\ \vdots \\ x^{(1)}(m) - x^{(1)}(m-1) \end{bmatrix} = \begin{bmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(m) & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{16}$$

By using the least squares to estimate the model in equation (16), the parameters which also satisfies as in (15) are

$$B = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}_{(m-1)\times m}$$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{m\times m} \begin{bmatrix} xx(1) & -1 \\ xx(2) & 0 \\ \vdots & \vdots \\ xx(m) & 0 \end{bmatrix}_{m\times 2}$$

$$= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}_{(m-1)\times m}$$

$$\begin{bmatrix} w_1 & 0 & \cdots & 0 \\ w_1 & w_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ w_1 & w_2 & \cdots & w_m \end{bmatrix}_{m\times m} \begin{bmatrix} x^{(0)}(1) & -\frac{1}{w_1} \\ x^{(0)}(2) & 0 \\ \vdots & \vdots \\ x^{(0)}(m) & 0 \end{bmatrix}_{m\times 2}$$

$$Y = [x^{(1)}(2) - x^{(1)}(1), x^{(1)}(3) - x^{(1)}(2), \ldots, \\ x^{(1)}(m) - x^{(1)}(m-1)]^T.$$

## IV. Fractional Accumulation GM $(1, 1)$ Predictor Theory

### A. Prediction Theory

Suppose the original data and the fractional-order accumulation generated sequence are $(x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(m))$, $(x^{(1)}(1), x^{(1)}(2), \cdots, x^{(1)}(m))$ and $(z^{(1)}(1), z^{(1)}(2), \cdots, z^{(1)}(m))$. $z^{(1)}(i) = [x^{(1)}(i) + x^{(1)}(i+1)]/2$, $i = 1, 2, \cdots, m-1$. For $0 < k \leq 1$, the estimated data from the model can be obtained by (12) and $\hat{x}^{(0)} = (\hat{x}^{(0)}(1), \hat{x}^{(0)}(2), \cdots, \hat{x}^{(0)}(m))$. The estimated prediction data $x^{(0)}(m+1)$ can be obtained from the following equation.

$$\hat{x}^{(0)}(m+1) = x^{(1)}(m+1) - \sum_{j=1}^{m} \frac{\Gamma(m+1-j+k)}{\Gamma(m-j+2)\Gamma(k)} \hat{x}^{(0)}(j) \tag{17}$$

The relative error is defined as

$$\begin{aligned} \Delta &= \left| \frac{\hat{x}^{(0)} - x^{(0)}}{x^{(0)}} \right| \times 100\% \\ &= \Bigg( \left| \frac{\hat{x}^{(0)}(1) - x^{(0)}(1)}{x^{(0)}(1)} \right| \times 100\%, \left| \frac{\hat{x}^{(0)}(2) - x^{(0)}(2)}{x^{(0)}(2)} \right| \\ &\quad \times 100\%, \cdots, \left| \frac{\hat{x}^{(0)}(m) - x^{(0)}(m)}{x^{(0)}(m)} \right| \times 100\% \Bigg) \end{aligned} \tag{18}$$

and $\overline{\Delta} = \frac{1}{m} \sum_{i=1}^{m} \Delta_i$ is the average relative error for the series modeling.

GA is a stochastic technique and popularly used in the optimization problems. It is inspired by the natural genetics and biological evolutionary process. Reproduction, crossover and mutation are three basic operators used to manipulate the genetic composition of a population. GA evaluates a population and generates a new one iteratively with each successive population (generation) [33]. In this paper, the fitness function is the minimization of the average model error and $\min_k \overline{\Delta} = \frac{1}{m} \sum_{i=1}^{m} \Delta_i$.

### B. Modified Predictor Model

The model residual sequence can be written as

$$\begin{aligned} \varepsilon^{(0)} &= (\varepsilon^{(0)}(2), \varepsilon^{(0)}(3), \cdots, \varepsilon^{(0)}(m))^{\mathrm{T}} \\ &= (x^{(0)}(2) - \hat{x}^{(0)}(2), x^{(0)}(3) - \hat{x}^{(0)}(3), \cdots, \\ &\quad x^{(0)}(m) - \hat{x}^{(0)}(m))^T \end{aligned}$$

The periodical feature hidden in the residual series can be extracted by the Fourier series and it is

$$\begin{aligned} &\varepsilon^{(0)}(j) = 0.5a_0 + \sum_{i=1}^{z} \left[ a_i \cos \frac{2\pi i}{T} j + b_i \sin \frac{2\pi i}{T} j \right] \\ &j = 2, 3, \cdots, m, T = m-1, z = [(m-1)/2] - 1 \end{aligned} \tag{19}$$

where $z$ is the integral part of $(m-1)/2 - 1$. It can be also written in the matrix form as below [25], [34].

$$\begin{bmatrix} \varepsilon^{(0)}(2) \\ \varepsilon^{(0)}(3) \\ \vdots \\ \varepsilon^{(0)}(m) \end{bmatrix} = \begin{bmatrix} 0.5 & \cos\frac{4\pi}{T} & \sin\frac{4\pi}{T} & \cdots & \cos\frac{4\pi z}{T} & \sin\frac{4\pi z}{T} \\ 0.5 & \cos\frac{6\pi}{T} & \sin\frac{6\pi}{T} & \cdots & \cos\frac{6\pi z}{T} & \sin\frac{6\pi z}{T} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0.5 & \cos\frac{2m\pi}{T} & \sin\frac{2m\pi}{T} & \cdots & \cos\frac{2m\pi z}{T} & \sin\frac{2m\pi z}{T} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ \vdots \\ a_z \\ b_z \end{bmatrix} \tag{20}$$

If

$$P = \begin{bmatrix} 0.5 & \cos\frac{4\pi}{T} & \sin\frac{4\pi}{T} & \cdots & \cos\frac{4\pi z}{T} & \sin\frac{4\pi z}{T} \\ 0.5 & \cos\frac{6\pi}{T} & \sin\frac{6\pi}{T} & \cdots & \cos\frac{6\pi z}{T} & \sin\frac{6\pi z}{T} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0.5 & \cos\frac{2m\pi}{T} & \sin\frac{2m\pi}{T} & \cdots & \cos\frac{2m\pi z}{T} & \sin\frac{2m\pi z}{T} \end{bmatrix},$$

$C = \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ \vdots \\ a_z \\ b_z \end{bmatrix}$, then the matrix can be written as $\varepsilon^{(0)} = PC$.

Based on the least square method, $\hat{C} = (P^T P)^{-1} P^T \varepsilon^{(0)}$. Therefore, the estimated residual error can be given as

$$\begin{aligned} &\hat{\varepsilon}^{(0)}(j) = 0.5\hat{a}_0 + \sum_{i=1}^{z} \left[ \hat{a}_i \cos \frac{2\pi i}{T} j + \hat{b}_i \sin \frac{2\pi i}{T} j \right] \\ &j = 2, 3, \cdots, m, T = m-1, z = \frac{[m-1]}{2} - 1 \end{aligned} \tag{21}$$

The modified model output $\hat{x}_F^{(0)}(j)$ by Fourier series is

$$\hat{x}_F^{(0)}(j) = \hat{x}^{(0)}(j) + \hat{\varepsilon}^{(0)}(j) \tag{22}$$

The process of calculating modified optimal fractional accumulation GM $(1,1)$ is:

(1)

1) Use GA to find the optimal fractional-order model with the minimization of the average relative error for modeling;
2) Analyze the estimated residual error using Fourier series;
3) Obtain the modified prediction model;
4) Analyze the model and the prediction results.

### C. State Space Model for Prediction

For the data sequence $x(j)$ $j=1,2,\cdots,m$, suppose we obtain the optimal fractional accumulation data sequence $x^{(1)}$ by GA, the fractional-order accumulation GM $(1,1)$ can be also represented by the state variable and the form is

$$\begin{aligned} x^{(1)}(n+1) &= A^{(1)}x^{(1)}(n)+B^{(1)}u(n) \\ y^{(1)}(n+1) &= x^{(1)}(n+1) \end{aligned} \tag{23}$$

where $x^{(1)}$ is the state variable vector and

$$x^{(1)}(n)=\sum_{j=1}^{n} {}^{(k)}x^{(0)}(j)=\sum_{j=1}^{n}\frac{\Gamma(n-j+k)}{\Gamma(n-j+1)\Gamma(k)}x(j),$$

$y$ is the output variable, $u$ is the input variable and equal to 1. Compared with the basic fractional accumulation GM $(1,1)$, when the model form is $x^{(1)}(i+1)-x^{(1)}(i)+ax^{(1)}(i+1)=b$, $A^{(1)}=1/(1+a), B^{(1)}=b/(1+a)$. When the model form is $x^{(1)}(i+1)-x^{(1)}(i)+az^{(1)}(i+1)=b$, $A^{(1)}=(2-a)/(2+a), B^{(1)}=b/(2+a)$. For $n<m$ and $n\geq m$, the state space model becomes a constructing and predicting model equation respectively.

As in (23), it can be also written as

$$\sum_{j=1}^{n+1} {}^{(k)}x^{(0)}(j)=A^{(1)}\sum_{j=1}^{n} {}^{(k)}x^{(0)}(j)+B^{(1)}u(n)$$

and

$$x^{(0)}(n+1)=A^{(0)}x^{(0)}(n)+B^{(0)}u(n)+\omega(n) \tag{24}$$

where, $A^{(0)}=A^{(1)}-\frac{k}{\Gamma(2)}, B^{(0)}=B^{(1)}, \omega(n)=\sum_{j=1}^{n-1}\frac{A^{(1)}\Gamma(n-j+k)\Gamma(n-j+2)-\Gamma(n-j+1)\Gamma(n-j+k+1)}{\Gamma(n-j+1)\Gamma(n-j+2)\Gamma(k)}x^{(0)}(j)$.

The solution of (23) can be obtained using irritation method, and

$$x^{(1)}(n)=\Phi(n,1)x(1)+\sum_{j=1}^{n-1}\Phi(n,j+1)B^{(1)}(j)u(j),$$

where, $\Phi(n,p)=A^{(1)}(n-1)A^{(1)}(n-2)...A^{(1)}(p)$, $\Phi(p,p)=I$. For $n<m$, take a difference operation on the left and right side in (23), then the difference of the state space equation can be written as:

$$\begin{aligned} \Delta x^{(1)}(n+1) &= A^{(1)}\Delta x^{(1)}(n) \\ y^{(1)}(n+1)-y^{(1)}(n) &= \Delta x^{(1)}(n+1) \\ &= A^{(1)}(x^{(1)}(n)-x^{(1)}(n-1)). \end{aligned}$$

The modeling process in (23) can be rewritten as

$$\begin{aligned} \begin{bmatrix}\Delta x^{(1)}(n+1)\\ y^{(1)}(n+1)\end{bmatrix} &= \begin{bmatrix}A^{(1)} & O_m^T\\ A^{(1)} & 1\end{bmatrix}\begin{bmatrix}\Delta x^{(1)}(n)\\ y^{(1)}(n)\end{bmatrix} \\ y^{(1)}(n) &= \begin{bmatrix}O_m & 1\end{bmatrix}\begin{bmatrix}\Delta x^{(1)}(n)\\ y^{(1)}(n)\end{bmatrix} \end{aligned},$$

where $O_m=\begin{bmatrix}0 & 0 & \cdots & 0\end{bmatrix}$. Define new variables $y(k)$ and $z(k)=\begin{bmatrix}\Delta x^{(1)}(k)\\ y^{(1)}(k)\end{bmatrix}$. It can be also given as

$$\begin{aligned} z(k+1) &= Az(k) \\ y(k) &= Cz(k) \end{aligned},$$

where $A=\begin{bmatrix}A^{(1)} & O_m^T\\ A^{(1)} & 1\end{bmatrix}$, $C=\begin{bmatrix}O_m & 1\end{bmatrix}$.

The characteristic equation of A can be calculated as

$$\begin{aligned} \det(\lambda I-A) &= \det\begin{bmatrix}\lambda I-A^{(1)} & -O_m^T\\ -A^{(1)} & \lambda I-1\end{bmatrix} \\ &= (\lambda-1)\det(\lambda I-A^{(1)}). \end{aligned}$$

The future state variables for the prediction can be calculated from the following equation. That is $Y=Fz(k)$, where $F=\begin{bmatrix}A\\ A^2\\ \dots\\ A^h\end{bmatrix}$ and $h$ is number of samples predicted for the future state variables.

Base on the Fourier series, the modified state space model for prediction is given as

$$\begin{aligned} \hat{x}(n+1) &= \hat{A}\hat{x}(n)+\hat{B}u(n)+e(n+1) \\ \hat{y}(n+1) &= \hat{x}(n+1) \end{aligned} \tag{25}$$

where,

$$e(n{+}1){=}0.5\hat{a}_0{+}\sum_{i=1}^{z}\left[\hat{a}_i\cos\frac{2\pi i}{T}(n{+}1){+}\hat{b}_i\sin\frac{2\pi i}{T}(n{+}1)\right].$$

## V. Fractional-order Grey Model

The fractional form for continuous data of GM $(1,1)$ model is defined as GM $(\alpha,1)$, and it is

$$\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha}+ax=b \tag{26}$$

where $a$ and $b$ are referred as the development coefficient and grey action quantity, respectively, the fractional-order $\alpha$ should be more than zero and the number of variables is one. It can also be written as

$$D_t^\alpha x+ax=b, \quad \text{or} \quad \frac{1}{b}D_t^\alpha x+\frac{a}{b}x=1.$$

The state space representation for GM $(\alpha,1)$ is

$$\begin{aligned} D_t^\alpha x(t) &= Ax(t)+Bu(t) \\ y(t) &= x(t) \end{aligned}$$

where $A,B,u(t)$ are the state, input and output vectors of the system and $A=-a, B=b, u(t)=1$.

In order to simplify the problem, here, we only consider the zero initial condition. The model solution can be obtained by using Laplace transform $X(s)=\frac{b}{s(s^{\alpha}+a)}$ and

$$x(t)=b\varepsilon_0(t,-a;\alpha,\alpha+1)=bt^{\alpha}E_{\alpha,\alpha+1}(-at^{\alpha}),$$

where

$$\varepsilon_k(t,\lambda;\mu,v)=t^{\mu k+v-1}E^{(k)}_{\mu,v}(\lambda t^{\mu})(k=0,1,2,\cdots)$$

is the Mittag-Leffler type introduced by Podlubny.

If $x^{(0)}=(x^{(0)}(1),\,x^{(0)}(2),\,\cdots,x^{(0)}(m))$ is a sequence of raw data which are usually non-negative values, its fractional-order accumulation generated sequence is given by $x^{(1)}=(x^{(1)}(1),\,x^{(1)}(2),\,\cdots,\,x^{(1)}(m))$, then $D^{\alpha}_t x^{(1)}+ax^{(1)}=b$ is referred to fractional-order whitenization or image equation of the GM$(\alpha,1)$.

By using forward difference with G-L definition, the following equation is obtained [35] which can be used for the discretization of continuous models.

$$\begin{aligned}D^{\alpha}x(kT)&\approx\frac{1}{T^{\alpha}}\sum_{i=0}^{k+1}(-1)^i\begin{pmatrix}\alpha\\ i\end{pmatrix}x((k+1-i)T)\\ &=\frac{1}{T^{\alpha}}(x((k+1)T)-\begin{pmatrix}\alpha\\ 1\end{pmatrix}x(kT)\\ &+\sum_{i=2}^{k+1}(-1)^i\begin{pmatrix}\alpha\\ i\end{pmatrix}x((k+1-i)T)\end{aligned}\tag{27}$$

## VI. Simulation Experiments

Consider a discrete sequence $x^{(0)}(j)=(1,2,3,4,5),j=1,2,\cdots,5$. The results of fractional-order weighted factor and accumulation generated sequence can be seen from Fig. 1 and Fig. 2. The fractional-order in the range of zero and one could model the system well compared with one-order generation and obey the new information priority rule.

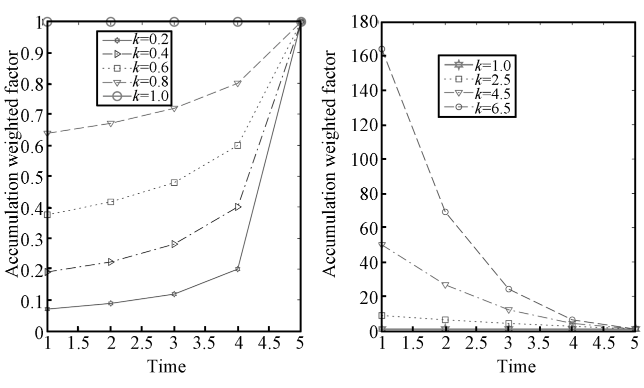


Fig. 1. Weighted factor with fractional-order less than and larger than one.

The original data of main indicators on internet development (based on number of users) taken at the end of years (2006-2012), from China statistical yearbook-2013, People's Republic of China National Bureau of Statistics, which can be seen in Table I. The model and error analysis results based on two models with fractional-order from 0.1 to 1.0 can be seen from Fig. 3 and Fig. 4, the prediction result and error analysis also show the predictor has better performance than the traditional GM$(1,1)$. The modified model result using Fourier series can be seen in Fig. 5 and Fig. 6. The optimal model and prediction result are obtained and can be seen in Fig. 7 and Fig. 8. The state space model for optimal order result is given in Table II. Error analysis of different fractional-order accumulations using Fourier series are given in Table III. The data of modified optimal fractional-order accumulation model can be seen in Table IV. In Table III, we can see that the average model and prediction error in the classical GM$(1,1)$ is larger than fractional-order accumulation ones. The modified model has better results than the traditional ones without using Fourier series in modeling and prediction. From Table IV, the modified GM using optimal fractional-order accumulation can obtain good performance compared with some classical GMs.

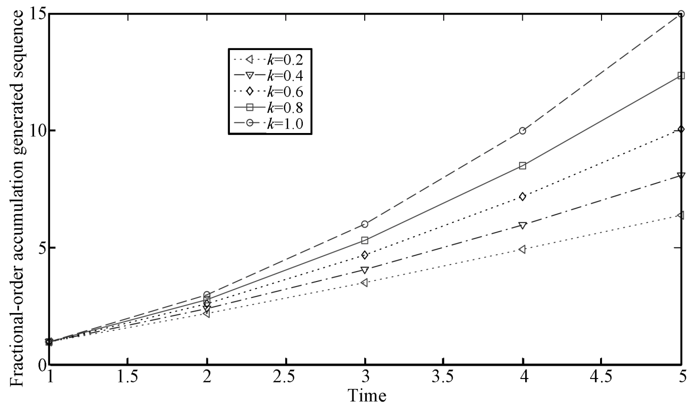


Fig. 2. Accumulation generated sequence with fractional-order.

Table I
Original Data of Number of Users Taken at the and of Years

| Year | 2006 | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 |
|---|---|---|---|---|---|---|---|
| Number of Internet Users ($10^6$ persons) | 137.0 | 210.0 | 298.0 | 384.0 | 457.3 | 513.1 | 564.0 |

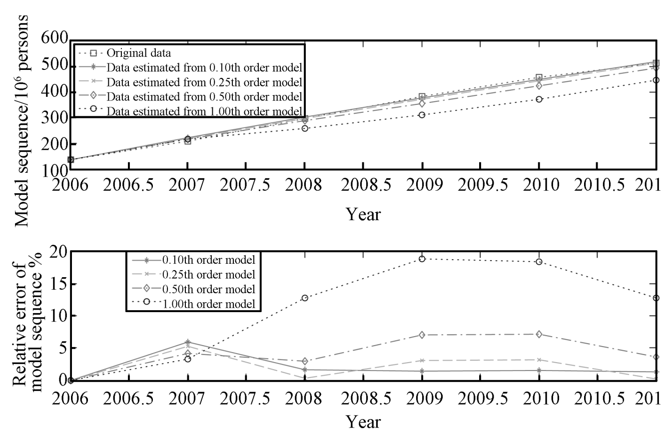


Fig. 3. Internet data modeling and error analysis using $x^{(1)}(k+1)-x^{(1)}(k)+ax^{(1)}(k+1)=b$.

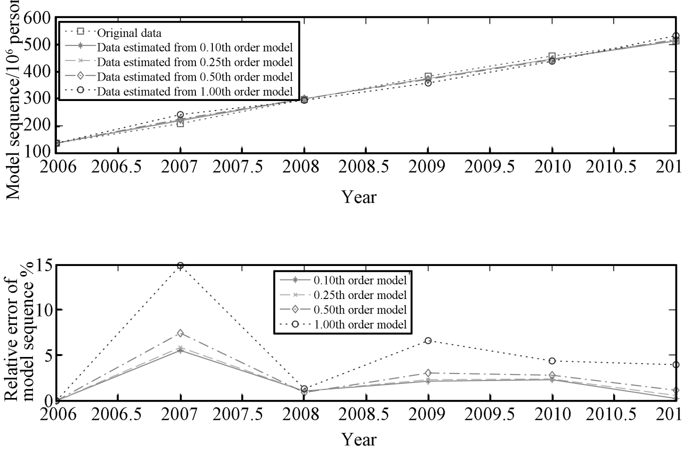


Fig. 4. Internet data modeling and error analysis using $x^{(1)}(k+1)-x^{(1)}(k)+az^{(1)}(k+1)=b$.

Table II
STATE SPACE MODEL FOR OPTIMAL ORDER

| GM model | Optimal order | $A^{(1)}$ | $B^{(1)}$ | $A$ | $C$ |
|---|---|---|---|---|---|
| $x^{(1)}(k+1)$ $-x^{(1)}(k)$ $+ax^{(1)}(k+1)=b$ | 0.2311 | 1.0112 | 114.9412 | $\begin{bmatrix}1.0112 & 0\\ 1.0112 & 1\end{bmatrix}$ | $\begin{bmatrix}0 & 1\end{bmatrix}$ |
| $x^{(1)}(k+1)$ $-x^{(1)}(k)$ $+az^{(1)}(k+1)=b$ | 1.6736 $\times 10^{-5}$ | 0.9427 | 46.1318 | $\begin{bmatrix}0.9427 & 0\\ 0.9427 & 1\end{bmatrix}$ | $\begin{bmatrix}0 & 1\end{bmatrix}$ |

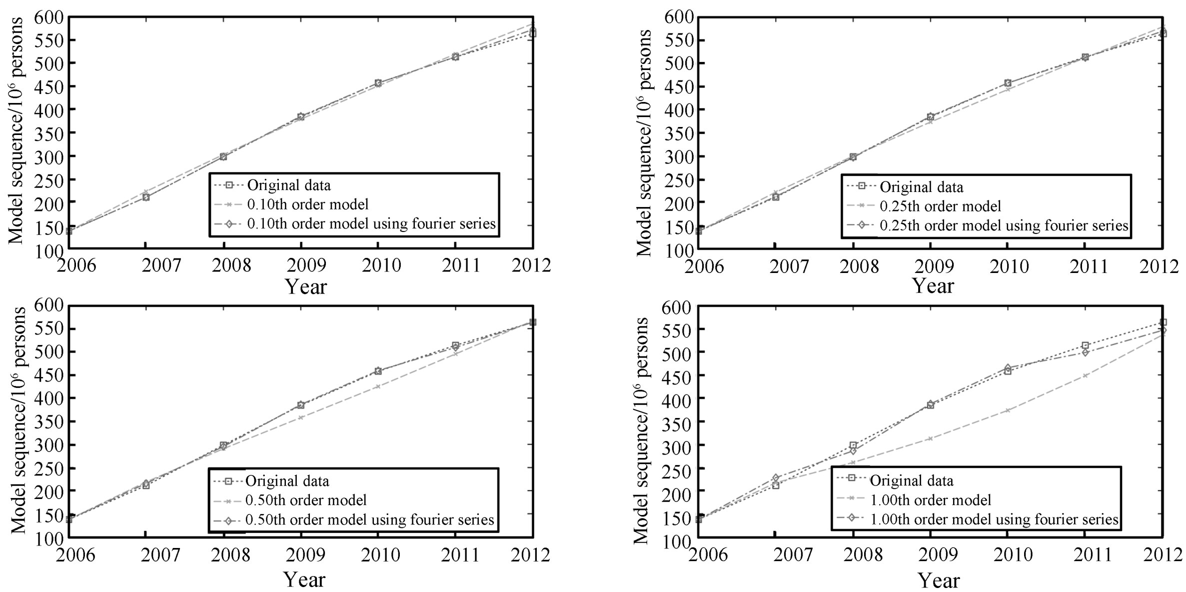


Fig. 5. Modified model output of internet data modeling using Fourier series analysis based on $x^{(1)}(k+1)-x^{(1)}(k)+ax^{(1)}(k+1)=b$.

Table III
ERROR ANALYSIS USING SERIES OF INTERNET DEVELOPMENT

| order | method | $x^{(1)}(k+1)-x^{(1)}(k)$ $+ax^{(1)}(k+1)=b$ | | $x^{(1)}(k+1)-x^{(1)}(k)$ $+az^{(1)}(k+1)=b$ | |
|---|---|---|---|---|---|
| | | $\Delta$ (%) | $\Delta_7$ (%) | $\Delta$ (%) | $\Delta_7$ (%) |
| 0.10 | common | 1.9236 | 3.7190 | 1.8549 | 2.5575 |
| | Fourier series | 0.1118 | 1.5075 | 0.1927 | 0.6877 |
| 0.25 | common | 1.9736 | 2.7617 | 2.0085 | 3.4366 |
| | Fourier series | 0.2456 | 1.0420 | 0.1693 | 1.3721 |
| 0.50 | common | 4.1491 | 0.5578 | 2.5374 | 5.7856 |
| | Fourier series | 0.8681 | 0.1029 | 0.2047 | 3.1108 |
| 0.75 | common | 7.2293 | 2.1057 | 3.4459 | 9.6037 |
| | Fourier series | 1.7676 | 1.5565 | 0.3419 | 5.8862 |
| 0.80 | common | 7.9322 | 2.6635 | 3.6752 | 10.5827 |
| | Fourier series | 1.9792 | 1.8657 | 0.3842 | 6.5982 |
| 0.90 | common | 9.4185 | 3.7836 | 4.3567 | 12.7859 |
| | Fourier series | 2.4345 | 2.4880 | 0.0168 | 8.2046 |
| 1.00 | common | 11.0073 | 4.8934 | 5.1734 | 15.3407 |
| | Fourier series | 2.9336 | 3.1033 | 0.6211 | 10.0767 |

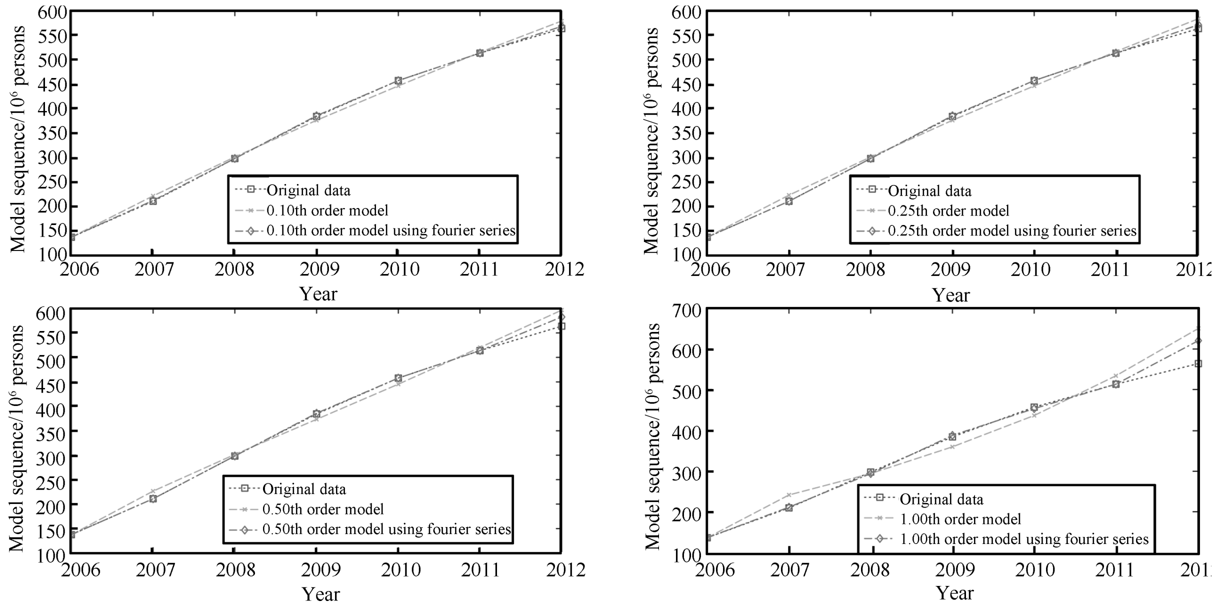


Fig. 6. Modified model output of internet data modeling using Fourier series analysis based on $x^{(1)}(k+1) - x^{(1)}(k) + az^{(1)}(k+1) = b$.

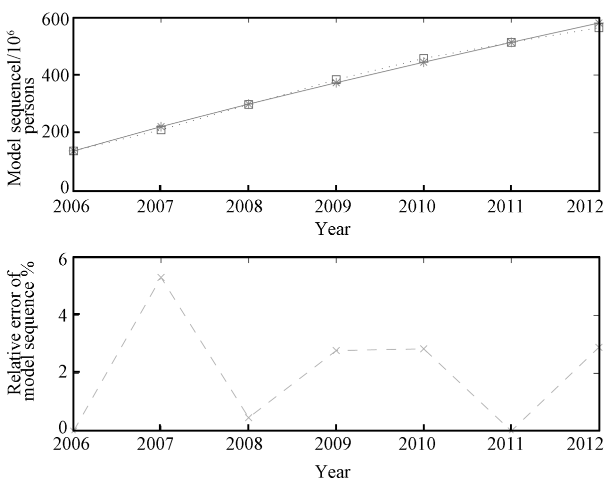


Fig. 7. Internet data optimal modeling and error analysis using $x^{(1)}(k+1) - x^{(1)}(k) + ax^{(1)}(k+1) = b$.

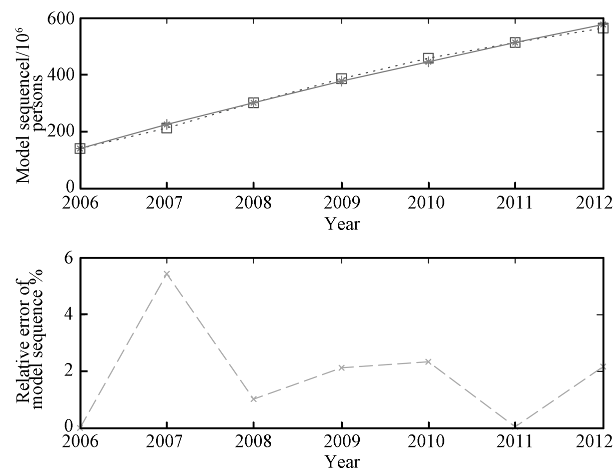


Fig. 8. Internet data optimal modeling and error analysis using $x^{(1)}(k+1) - x^{(1)}(k) + az^{(1)}(k+1) = b$.

Table IV

Optimal Model Analysis and Prediction Result Data Using GA by Fourier Series

| $x^{(1)}(k+1) - x^{(1)}(k)$ $+ax^{(1)}(k+1) = b$ | $\overline{\Delta}(\%)$ | Prediction value | $\Delta_7(\%)$ |
|---|---|---|---|
| GA method | 1.9044 | 580.3567 | 2.9001 |
| GA method by Fourier series | 0.2176 | 570.2651 | 1.1108 |
| One-order accumulation GM(1,1) | 11.0073 | 536.4013 | 4.8934 |
| $x^{(1)}(k+1) - x^{(1)}(k) + az^{(1)}(k+1) = b$ | $\overline{\Delta}(\%)$ | Prediction value | $\Delta_7(\%)$ |
| GA method | 1.8126 | 576.0692 | 2.1399 |
| GA method by Fourier series | 0.2240 | 565.9218 | 0.3407 |
| One-order accumulation GM(1,1) | 5.1734 | 650.5216 | 15.3407 |
| Model | $\overline{\Delta}(\%)$ | Prediction value | $\Delta_7(\%)$ |
| DGM(1,1) | 5.7740 | 653.7459 | 15.9124 |
| Model | $\overline{\Delta}(\%)$ | Prediction value | $\Delta_7(\%)$ |
| DGM(2,1) | 11.0738 | 618.3561 | 9.6376 |

From the above results, the model prediction performance can be greatly improved by Fouries series method, and the modified optimal predictor could describe the system more precisely than the traditional GM(1,1) in model construction and prediction. The difference of $x^{(1)}(k+1) - x^{(1)}(k) + ax^{(1)}(k+1) = b$ and $x^{(1)}(k+1) - x^{(1)}(k) + az^{(1)}(k+1) = b$ is that the latter model considers the neighbour information in the data modeling. From the optimal order analysis and other data results, when the model is different, the optimal fractional-order is also different for the same problem. Several other results also show the proposed method of fractional accumulation order using GA by Fourier series can improve the model performance greatly and can be perfectly put into the application of predictor design and model construction.

## VII. Conclusion

Fractional calculus studies the possibility of differentiation and integration with arbitrary real or complex orders of the differential operator. The fractional-order accumulation model can mine the data information more precisely than the classical order, which can reduce the randomness of the original data and exhibit better performance. The GM$\,(1,1)$ predictor based on the fractional-order accumulation has more degrees of freedom and better performance compared with the traditional one-order GM. The modified optimal fractional-order accumulation method can be used to ameliorate the model performance and improve the accuracy of predictor. The GM$\,(\alpha,1)$ application scope will also be enlarged with more freedom. The proposed method can be widely used in data processing, modeling theory, multi-step prediction, predictor design and related fields. The discrete and continuous forms for GM by fractional calculus, other forms of GMs by fractional accumulation, for example, delay model, power exponent model and feedback model, which also need our further researches.

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