Stability Analysis, Chaos Control of Fractional Order Vallis and El-Nino Systems and Their Synchronization

Subir Das and Vijay K Yadav

Abstract—**In this article the authors have studied the stability analysis and chaos control of the fractional order Vallis and El-Nino systems. The chaos control of these systems is studied using nonlinear control method with the help of a new lemma for Caputo derivative and Lyapunov stability theory. The synchronization between the systems for different fractional order cases and numerical simulation through graphical plots for different particular cases clearly exhibit that the method is easy to implement and reliable for synchronization of fractional order chaotic systems. The comparison of time of synchronization when the systems pair approaches from standard order to fractional order is the key feature of the article.**

Index Terms—**El-Nino system, fractional derivative, nonlinear control method, stability analysis, synchronization, Vallis systems.**

I. INTRODUCTION

THE chaotic system is a nonlinear deterministic system which possess complex dynamical behaviors which are extremely sensitive to initial conditions and having bounded trajectories in the phase space. The study of dynamic behavior in nonlinear fractional order systems has become an interesting topic to the scientists and engineers. Fractional calculus is playing an important role for the analysis of nonlinear dynamical systems. Through fractional calculus approach many systems in interdisciplinary fields can be described by the fractional differential equation such as dielectric polarization, viscoelastic system, electrode-electrolyte polarization and electronic wave [1]−[4]. Another importance of fractional calculus is that it provides an excellent tool for the description of memory and hereditary properties, for which it is used in various physical areas of science and engineering such as material science [5], fluid mechanics [6], colored noise, biological modeling [7], [8], etc.

Effect of chaos in nonlinear dynamics is studied during last few decades by the researchers from different parts of the world. This effect is most common and has been detected in a number of dynamical systems of various types of physical nature. In practice it is usually undesirable and restricts the operating range of many mechanical and electrical devices. This type of control of dynamical system has attracted a great deal of attention by the researchers in society of engineering. The chaos control of systems can be divided into two categories, first one is to suppress the chaotic dynamical behaviors and second one is to generate or enhance chaos in nonlinear systems known as chaotincation or anti-control of chaos. So far various types of methods and techniques have been proposed for control of chaos such as feedback and non-feedback control [9]−[11], adaptive control [12], [13] and backstepping method [14] etc. Synchronization of two dynamical systems is the phenomenon where one dynamical system behaves according to the behavior of the other dynamical system. In chaos synchronization, two or more chaotic systems are coupled or one chaotic system drives another system. L. M. Pecora and T. L. Carroll [15] were first to introduce a method to synchronize drive and response systems of two identical or non-identical systems with different initial conditions.

In this manuscript, the authors have studied the chaos control and stability analysis of Vallis and El-Nino systems with fractional order, and also the synchronization between the considered systems. A nonlinear control method is used for chaos control of fractional order Vallis and El-Nino systems, and also during their synchronization. Both the systems were proposed by Vallis in 1986 for the description of temperature fluctuations in the western and eastern parts of equatorial ocean, which have a strong influence on the Earth's global climate. The first model Vallis system does not allow trade winds, whereas the second model El-Nino system describes the nonlinear interactions of the atmosphere, and trade winds in the equatorial part of pacific ocean. The main feature of this article is the study of time of synchronization between the systems through numerical simulation for different particular cases as systems' pair approaches fractional order from integer order.

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II. SOME PRELIMINARIES AND STABILITY CONDITION

A. Definitions and Lemma

Definition 1 [16]: The Caputo derivative for fractional order q is defined as

$$ {}_a^cD_t^q\phi(t) = \frac{1}{\Gamma(n-q)}\int_a^t \frac{\phi^{(n)}(\tau)}{(t-\tau)^{q+1-n}}\,d\tau, \quad t > a $$

where $q \in \mathbb{R}^+$ on the half axis $\mathbb{R}^+$ and $n = \min\{k \in \mathbb{N}/k > q\}$, $q > 0$.

Lemma 1 [17]: Let $x(t) \in \mathbb{R}$ be a continuous and derivable function. Then for any time instant $t \geq t_0$,

$$\frac{1}{2}{}_{t_0}^{c}D_t^q x^2(t) \leq x(t)_{t_0}^{c}D_t^q x(t) \quad \forall\, q \in (0, 1).$$

Definition 2 [18]: An equilibrium point E of a system is called a saddle point of index 1 if the Jacobian matrix at point E has one eigenvalue with a non-negative real part (i.e., unstable).

Definition 3: An equilibrium point E of a system is called a saddle point of index 2 if the Jacobian matrix at point E has two unstable eigen values.

The scrolls are generated only around the saddle points of index 2. Saddle point of index 1 is responsible only for connecting scrolls.

B. Stability of the System

Consider a fractional order dynamical system as

$$\begin{aligned} D_t^q x(t) &= f_1(x,\, y,\, z) \\ D_t^q y(t) &= f_2(x,\, y,\, z) \\ D_t^q z(t) &= f_3(x,\, y,\, z) \end{aligned} \tag{1}$$

where $q \in (0,\, 1)$ and D_t^q is the Caputo derivative. The Jacobian matrix at equilibrium points of the above system is

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}. \tag{2}$$

Theorem 1 [19], [20]: The system (1) is locally asymptotically stable if all the eigenvalues of the Jacobian matrix at its equilibrium point satisfy the condition

$$|\arg(\lambda)| > \frac{q\pi}{2}. \tag{3}$$

III. Design of Controller for Fractional Order Chaotic System Using Nonlinear Control Method

Consider the fractional order chaotic system as the master system as

$$D_t^q x = Px + Qf(x) \tag{4}$$

where $0 < q \leq 1$ is the order of the fractional time derivative, $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$ is the state vector, P and Q are the $n \times n$ matrices consisting of the system parameters and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function of the system.

Consider another fractional order chaotic system as a slave system described as

$$D_t^q y = P_1 y + Q_1 g(y) + u(t) \tag{5}$$

where $y = [y_1, y_2, \ldots, y_n]^T \in \mathbb{R}^n$ is the state vector of the system, P_1 and Q_1 are the $n \times n$ matrices of the system parameters, $g : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear part of the function of the system and $u(t)$ is the controller of the system (5).

During synchronization, defining the error as $e = y - x$, the error dynamical system is obtained as

$$D_t^q e = P_1 e + Q_1 g(y) + (P_1 - P)x - Qf(x) + u(t). \tag{6}$$

During the synchronization, our aim is to find the appropriate feedback controller $u(t)$, so that we may stabilize the error dynamics (6) in order to get $\lim_{t\to\infty} \|e(t)\| = 0, \forall\, e(0) \in \mathbb{R}^n$.

Now, we define the following Lyapunov function as $V(e) = \frac{1}{2}e^T e$, whose q-th order fractional derivative w. r. to t is (using Lemma 1)

$$\begin{aligned} \frac{d^q V(e)}{dt^q} &= \frac{1}{2}\frac{d^q(e^{\mathrm{T}}e)}{dt^q} = \frac{1}{2}\frac{d^q}{dt^q}\left(e_1^2 + e_2^2 + \cdots + e_n^2\right) \\ &\leq \left(e_1\frac{d^q e_1}{dt^q} + e_2\frac{d^q e_2}{dt^q} + \cdots + e_n\frac{d^q e_n}{dt^q}\right). \end{aligned} \tag{7}$$

Substituting the values of $\frac{d^q e_1}{dt^q}, \frac{d^q e_2}{dt^q}, \ldots, \frac{d^q e_n}{dt^q}$ and choosing appropriate control function $u(t)$, the q-th order derivative of the Lyapunov function $V(e)$ becomes negative i.e., $\frac{d^q V(e)}{dt^q} < 0$, which helps to get synchronization between the systems (4) and (5).

IV. System Description and Its Stability

A. Fractional Order Vallis System

The Vallis model [21], [22] is described by

$$\begin{aligned} \frac{dx}{dt} &= \mu y - ax \\ \frac{dy}{dt} &= xz - y \\ \frac{dz}{dt} &= 1 - xy - z \end{aligned} \tag{8}$$

where x is the speed of water molecules on the surface of ocean, $y = (T_w - T_e)/2$, $z = (T_w + T_e)/2$, T_w and T_e are temperatures accordingly in western and eastern parts of ocean, μ and a are positive parameters.

The fractional order Vallis system can be described as

$$\begin{aligned} \frac{d^q x}{dt^q} &= \mu y - ax \\ \frac{d^q y}{dt^q} &= xz - y \\ \frac{d^q z}{dt^q} &= 1 - xy - z, \quad 0 < q < 1. \end{aligned} \tag{9}$$

1) Equilibrium Points and Stability: To find the equilibrium points of the system (9), we have

$$D_t^q x = D_t^q y = D_t^q z = 0$$

where $D_t^q = \frac{d^q}{dt^q}$.

The equilibrium points are obtained as

$$\begin{aligned} E_1 &= (0,\ 0,\ 1) \\ E_{2,\,3} &= \left(\pm\sqrt{\frac{(\mu - a)}{a}},\ \pm\frac{\sqrt{a(\mu - a)}}{\mu},\ \frac{a}{\mu}\right). \end{aligned}$$

For convenience the point E_1 is shifted to the point of origin through the transformation $z \to z + 1$ and the system (9) reduces to

$$\frac{d^q x}{dt^q} = \mu y - ax$$
$$\frac{d^q y}{dt^q} = xz + x - y$$
$$\frac{d^q z}{dt^q} = -xy - z \tag{10}$$

where $0 < q < 1$. For the parameters $\mu = 121$ and $a = 5$ and the initial condition $(0.1, 1.2, 0.5)$, the trajectories of the Vallis system are depicted through Figs. 1 (a)−1 (d) for fractional order $q = 0.97$. Again for the same parameters' values and initial conditions the Vallis system shows chaotic behavior at the lowest fractional order $q = 0.981$, the trajectories of which are described through Figs. 2 (a)−2 (d).

The equilibrium points of the system (10) are $E_1 = (0,\ 0,\ 0)$, $E_2 = (4.8166,\ 0.1990,\ -0.9586)$ and $E_3 = (-4.8166,\ -0.1990,\ -0.9586)$.

The Jacobian matrix of the Vallis system (10) at the equilibrium point $\bar{E}(\bar{x},\ \bar{y},\ \bar{z})$ is

$$J(\bar{E}) = \begin{bmatrix} -a & \mu & 0 \\ \bar{z}+1 & -1 & \bar{x} \\ -\bar{y} & -\bar{x} & -1 \end{bmatrix}. \tag{11}$$

Putting the values of $a = 5$ and $\mu = 121$, the characteristic polynomial of the above Jacobian matrix will be

$$P(\lambda) = \lambda^3 + 7\lambda^2 - (-\bar{x}^2 + 121\bar{z} + 110)\lambda - 121\bar{z} + 5\bar{x}^2 + 121\bar{x}\bar{y} - 116. \tag{12}$$

At the equilibrium point $E_1 = (0,\ 0,\ 0)$, (12) becomes

$$P(\lambda) = \lambda^3 + 7\lambda^2 - 110\lambda - 116. \tag{13}$$

The eigenvalues of the equation (13) are $\lambda_1 = -14.1803$, $\lambda_2 = 8.1803$, $\lambda_3 = -1.0000$.

It is seen that the equilibrium point E_1 is a saddle point of index 1 and from Definition 2 it is unstable for $0 < q < 1$.

At the equilibrium point $E_2 = (4.8166,\ 0.1990,\ -0.9586)$, the equation (12) becomes

$$P(\lambda) = \lambda^3 + 7\lambda^2 + 29.1902\lambda + 231.9676. \tag{14}$$

The eigenvalues of (14) are $\lambda_1 = -7.3331$, $\lambda_{2,3} = 0.1666 \pm 5.6219i$. The equilibrium point E_2 is the saddle point of index 2 (Definition 2). E_2 is stable for $0 < q < 0.981$. Similarly the equilibrium point $E_3 = (-4.8166, -0.1990,\ -0.9586)$ is also stable for $0 < q < 0.981$.

2) Control of Chaos Using Nonlinear Control Method: Let the fractional order Vallis system is taken as a controlled system with control functions $u_i(t),\ i = 1,\ 2,\ 3$ to stabilize unstable periodic orbit or fixed point as given in (10).

Let $(\bar{x},\ \bar{y},\ \bar{z})$ is the solution of the system (10), then we have

$$\frac{d^q \bar{x}}{dt^q} = \mu\bar{y} - a\bar{x}$$
$$\frac{d^q \bar{y}}{dt^q} = \bar{x}\bar{z} + \bar{x} - \bar{y}$$
$$\frac{d^q \bar{z}}{dt^q} = -\bar{x}\bar{y} - \bar{z}. \tag{15}$$

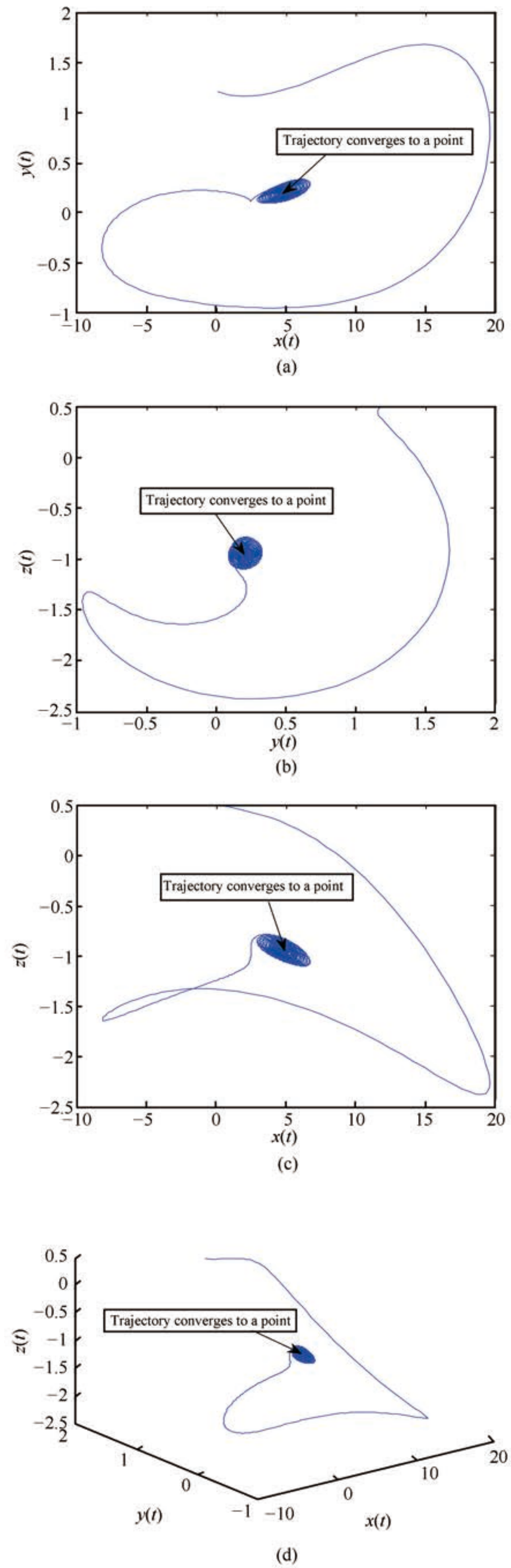


Fig. 1. Phase portraits of fractional order Vallis system for fractional order $q = 0.97$.

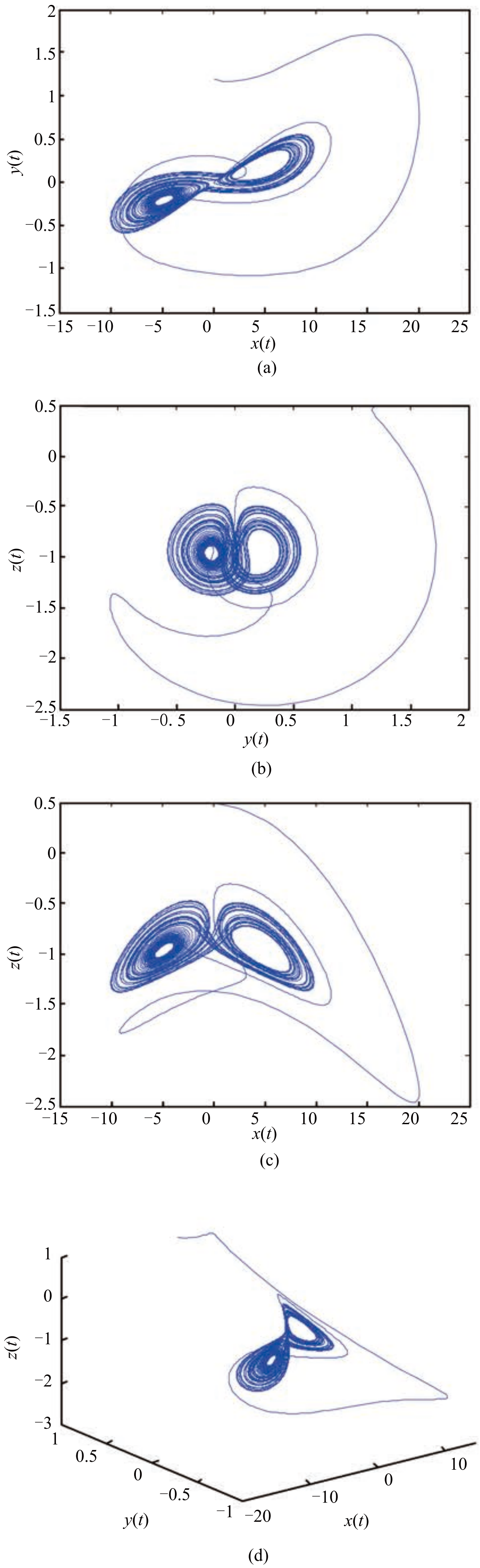


Fig. 2. Phase portraits of fractional order Vallis system for fractional order $q = 0.981$.

Defining error functions as $e_1 = x - \bar{x}$, $e_2 = y - \bar{y}$ and $e_3 = z - \bar{z}$, we obtain the following error system as

$$\begin{aligned}\frac{d^q e_1}{dt^q} &= \mu e_2 - a e_1 + u_1(t)\\ \frac{d^q e_2}{dt^q} &= e_1 - e_2 + xz - \bar{x}\bar{z} + u_2(t)\\ \frac{d^q e_3}{dt^q} &= -e_3 - xy + \bar{x}\bar{y} + u_3(t).\end{aligned} \tag{16}$$

To stabilize the error system, define the Lyapunov function as

$$V = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 + \frac{1}{2}e_3^2$$

whose q-th order fractional derivative is (from Lemma 1)

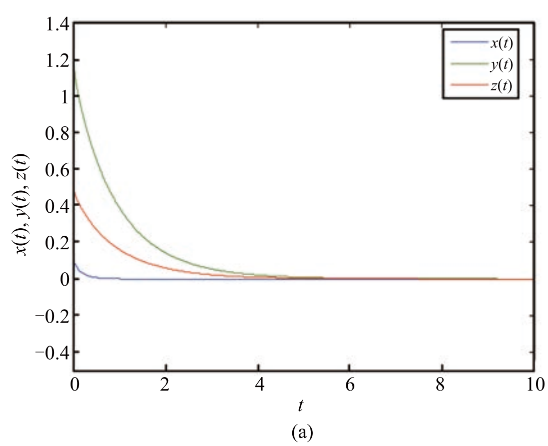
$$\begin{aligned}\frac{d^q V}{dt^q} &= \frac{1}{2}\frac{d^q e_1^2}{dt^q} + \frac{1}{2}\frac{d^q e_2^2}{dt^q} + \frac{1}{2}\frac{d^q e_3^2}{dt^q}\\ &\le e_1\frac{d^q e_1}{dt^q} + e_2\frac{d^q e_2}{dt^q} + e_3\frac{d^q e_3}{dt^q}\end{aligned}$$

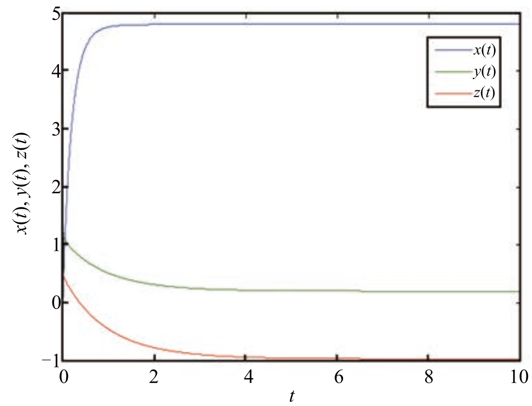
i.e.,

$$\begin{aligned}\frac{d^q V}{dt^q} \le\ & e_1[\mu e_2 - a e_1 + u_1(t)]\\ &+ e_2[e_1 - e_2 + xz - \bar{x}\bar{z} + u_2(t)]\\ &+ e_3[-e_3 - xy + \bar{x}\bar{y} + u_3(t)].\end{aligned}$$

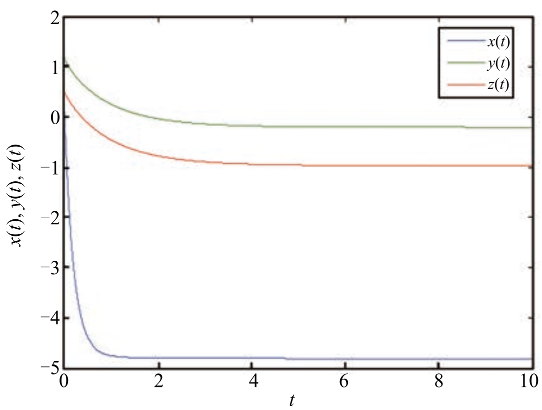
If we take $u_1(t) = -\mu e_2$, $u_2(t) = -e_1 - xz + \bar{x}\bar{z}$ and $u_3(t) = xy - \bar{x}\bar{y}$, it becomes $\frac{d^q V}{dt^q} \le -a e_1^2 - e_2^2 - e_3^2 < 0$. This shows that the trajectories $(x(t), y(t), z(t))$ converge to the point $(\bar{x}, \bar{y}, \bar{z})$.

3) Stabilizing the Points E_1, E_2 and E_3: It is clear from Figs. 3 (a)–3 (c) that at $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 0) = E_1$, the system (10) is stable at the point E_1 for the order $0 < q < 1$. Similarly for $(\bar{x}, \bar{y}, \bar{z}) = (4.8166, 0.1990, -0.9586) = E_2$ and $(\bar{x}, \bar{y}, \bar{z}) = (-4.8166, -0.1990, -0.9586) = E_3$, the system (10) is also stable for the order $0 < q < 1$. The plots of the control functions $u_1(t)$, $u_2(t)$, $u_3(t)$ used to stabilize the fractional order chaotic system are depicted through Fig. 3 (d), which clearly show that the chosen functions tend to zero as time approaches infinity at the equilibrium point E_1. It can be shown that the nature of the above functions at other two equilibrium points E_2 and E_3 are similar.





(b)



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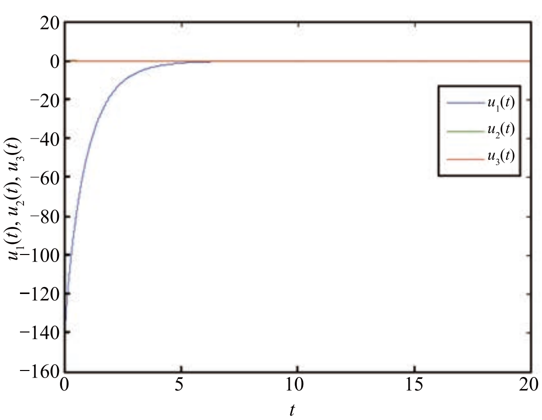


(d)

Fig. 3. Plots of $x(t)$, $y(t)$, $z(t)$ of the controlled system (10). (a) At equilibrium point E_1. (b) At the equilibrium point E_2. (c) At the equilibrium point E_3. (d) Plots of control functions $u_1(t)$, $u_2(t)$, $u_3(t)$ at E_1.

B. Fractional Order El-Nino System

El-Nino system is nonlinear and non-autonomous system represented by three differential equations as [21], [22]

$$\begin{aligned}\frac{dx}{dt}&=\mu'(y-z)-b(x-f(t))\\ \frac{dy}{dt}&=xz-y+c\\ \frac{dz}{dt}&=-xy-z+c\end{aligned}\tag{17}$$

where x, y and z are the speed of surface ocean current, temperature of water accordingly on western and eastern bounds of water pool respectively, $f(t)$ is the periodic function considering influence of trades winds.

Taking $f(t)=0$ to make an autonomous system as

$$\begin{aligned}\frac{dx}{dt}&=\mu'(y-z)-bx\\ \frac{dy}{dt}&=xz-y+c\\ \frac{dz}{dt}&=-xy-z+c.\end{aligned}\tag{18}$$

The fractional order El-Nino system is described as

$$\begin{aligned}\frac{d^qx}{dt^q}&=\mu'(y-z)-bx\\ \frac{d^qy}{dt^q}&=xz-y+c\\ \frac{d^qz}{dt^q}&=-xy-z+c,\quad 0<q<1.\end{aligned}\tag{19}$$

1) Equilibrium Points and Stability: To find the equilibrium points of the system (19), we have

$$\begin{aligned}&\mu'(y-z)-bx=0\\ &xz-y+c=0\\ &-xy-z+c=0.\end{aligned}$$

The equilibrium points are obtained as

$$P_1=(0,\ c,\ c)$$

$$P_2=\left(\sqrt{\frac{2\mu c}{b}-1},\ \frac{b+\sqrt{2\mu bc-b^2}}{2\mu},\ \frac{b-\sqrt{2\mu bc-b^2}}{2\mu}\right)$$

$$P_3=\left(-\sqrt{\frac{2\mu c}{b}-1},\ \frac{b-\sqrt{2\mu bc-b^2}}{2\mu},\ \frac{b+\sqrt{2\mu bc-b^2}}{2\mu}\right).$$

Making a shifting through $y\to y+c$ and $z\to z+c$, the system (19) will be reduced to the following form

$$\begin{aligned}\frac{d^qx}{dt^q}&=\mu'(y-z)-bx\\ \frac{d^qy}{dt^q}&=xz+xc-y\\ \frac{d^qz}{dt^q}&=-xy-xc-z.\end{aligned}\tag{20}$$

For the parameters $\mu'=83.6$, $b=10$ and $c=12$ and the initial condition $(-2,\ 3,\ 5)$, the El-Nino system shows chaotic behavior at $q=0.934$, the lowest fractional order (see Figs. 4 (a)$-$4 (d)). For the same values of parameters and initial conditions the trajectories of the system at $q=0.93$ are described through Figs. 5 (a)$-$5 (d).

The equilibrium points of the system (20) are calculated as

$$\begin{aligned}P_1&=(0,\ 0,\ 0)\\ P_2&=(14.1294,\ -11.0951,\ -12.7852)\\ P_3&=(-14.1294,\ -12.7852,\ -11.0951).\end{aligned}$$

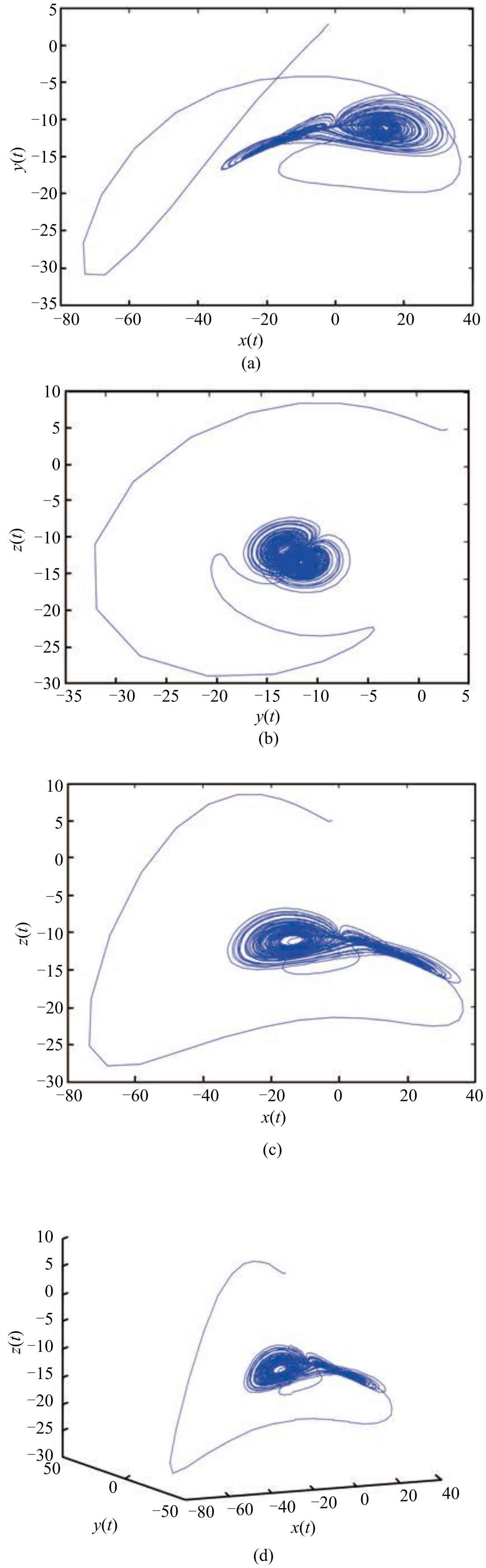


Fig. 4. Phase portraits of fractional order El-Nino system for fractional order $q = 0.934$.

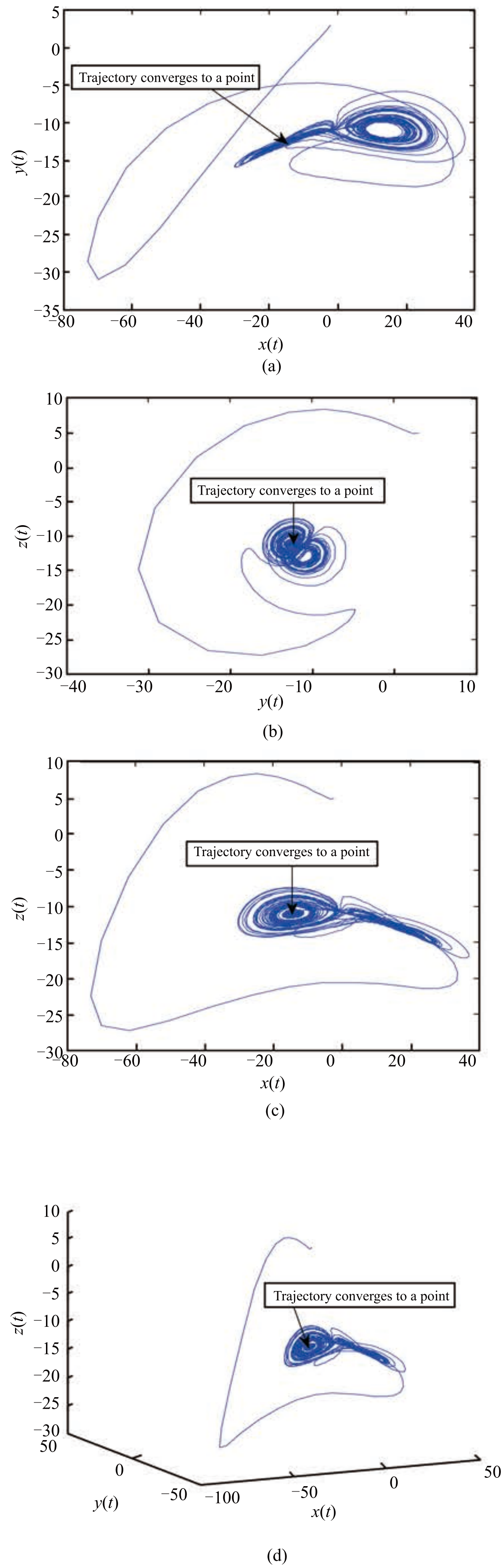


Fig. 5. Phase portraits of fractional order El-Nino system for fractional order $q = 0.93$.

The Jacobian matrix of the El-Nino system (20) at the equilibrium point $\bar{P}(\bar{x},\ \bar{y},\ \bar{z})$ is

$$J(\bar{P})=\begin{bmatrix} -b & \mu' & -\mu' \\ \bar{z}+c & -1 & \bar{x} \\ -\bar{y}-c & -\bar{x} & -1 \end{bmatrix}.$$

Putting the values of $\mu'=83.6$, $b=10$ and $c=12$, we obtain characteristic polynomial of the above Jacobian matrix as

$$\begin{aligned} P(\lambda)=&\ \lambda^3+12\lambda^2-(-\bar{x}^2+83.6\bar{y}+83.6\bar{z}+1985.40)\lambda \\ &+10\bar{x}^2-83.6\bar{y}-83.6\bar{z}+83.6\bar{x}\,\bar{y} \\ &-83.6\bar{x}\,\bar{z}-1996.40. \end{aligned}$$

At the equilibrium point $P_1=(0,\ 0,\ 0)$,

$$P(\lambda)=\lambda^3+12\lambda^2-1985.40\lambda-1996.40$$

which gives $\lambda_1=-50.5183$, $\lambda_2=39.5183$, $\lambda_3=-1.0000$.

Now P_1 is a saddle point of index 1 and from Definition 2 it is unstable for $0<q<1$. At the equilibrium point $P_2=(14.1294,\ -11.0951,\ -12.7852)$,

$$P(\lambda)=\lambda^3+12\lambda^2+210.6330\lambda+3992.7687$$

and thus the eigenvalues are $\lambda_1=-15.2956$, $\lambda_{2,\,3}=1.6478 \pm 16.0725i$. P_2 is the saddle point of index 2 (Definition 2). So P_2 is stable for $0<q<0.934$. Similarly at $P_3=(-14.1294,\ -12.7852,\ -11.0951)$, the eigenvalues are obtained as $\lambda_1=-15.2956$, $\lambda_{2,\,3}=1.6478\pm 16.0725i$, and this shows that P_3 is stable for $0<q<0.934$.

2) Control of Chaos Using Nonlinear Control Method: Consider the fractional order El-Nino system as a controlled system with control functions $u_1(t)$, $u_2(t)$ and $u_3(t)$, for stabilizing unstable periodic orbit and $(\bar{x},\ \bar{y},\ \bar{z})$ be the solution of the system (20) so that

$$\begin{aligned} \frac{d^q\bar{x}}{dt^q}&=\mu'(\bar{y}-\bar{z})-b\bar{x} \\ \frac{d^q\bar{y}}{dt^q}&=\bar{x}\bar{z}+\bar{x}c-\bar{y} \\ \frac{d^q\bar{z}}{dt^q}&=-\bar{x}\bar{y}-\bar{x}c-\bar{z}. \end{aligned} \tag{21}$$

Defining the error function $e(t)$ and Lyapunov function V as in Section IV-A for stabilizing the error system, we get the q-th order derivative of V as

$$\begin{aligned} \frac{d^qV}{dt^q}\le&\ e_1[\mu'(e_2-e_3)-be_1+u_1(t)] \\ &+e_2[ce_1-e_2+xz-\bar{x}\bar{z}+u_2(t)] \\ &+e_3[-ce_1-e_3-xy+\bar{x}\bar{y}+u_3(t)]. \end{aligned}$$

Taking $u_1(t)=-\mu'(e_2-e_3)$, $u_2(t)=-ce_1-xz+\bar{x}\bar{z}$ and $u_3(t)=ce_1+xy-\bar{x}\bar{y}$, we get $\frac{d^qV}{dt^q}\le -be_1^2-e_2^2-e_3^2<0$, which implies the trajectories $(x(t),\ y(t),\ z(t))$ converge to $(\bar{x},\ \bar{y},\ \bar{z})$.

3) Stabilizing the Points P_1, P_2, and P_3: It is seen from Figs. 6 (a)−6 (c) that at $P_1=(0,\ 0,\ 0)$, $P_2=(14.1294,\ -11.0951,\ -12.7852)$ and $P_3=(-14.1294,\ -12.7852,\ -11.0951)$, the system (20) is stable for the order $0<q<1$. Like previous system, the chosen control functions for this fractional order chaotic system converge to zero at all the equilibrium points P_1, P_2, P_3 as time approaches infinity. The plots at P_1 are shown through Fig. 6 (d).

V. Synchronization Between Fractional Order Vaallis and EL-Nino Systems Using Nonlinear Control Method

In this section to study the synchronization between fractional order Vallis and El-Nino systems, we consider the fractional order Vallis system as the master system as

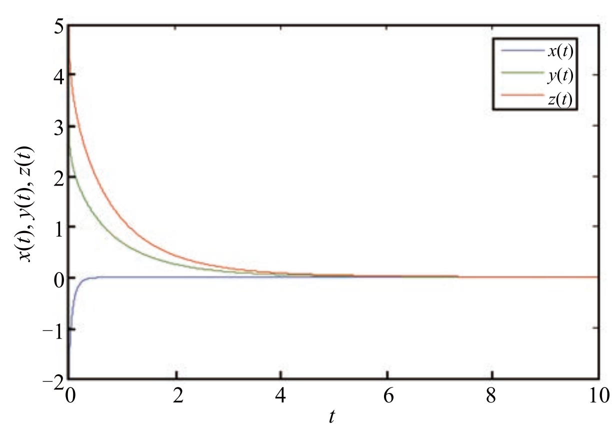
$$\begin{aligned} \frac{d^qx_1}{dt^q}&=\mu y_1-ax_1 \\ \frac{d^qy_1}{dt^q}&=x_1z_1+x_1-y_1 \\ \frac{d^qz_1}{dt^q}&=-x_1y_1-z_1 \end{aligned} \tag{22}$$

and the fractional order El-Nino system as slave system as

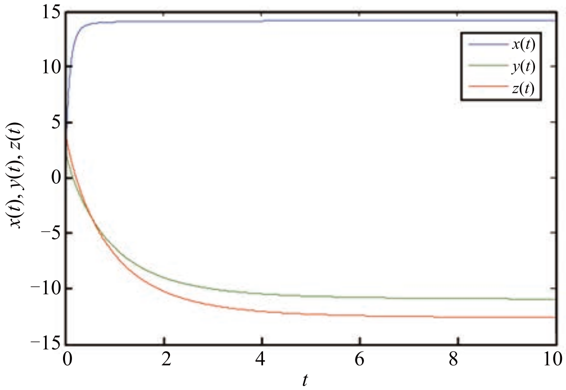
$$\begin{aligned} \frac{d^qx_2}{dt^q}&=\mu'(y_2-z_2)-bx_2+v_1(t) \\ \frac{d^qy_2}{dt^q}&=x_2z_2+x_2c-y_2+v_2(t) \\ \frac{d^qz_2}{dt^q}&=-x_2y_2-x_2c-z_2+v_3(t) \end{aligned} \tag{23}$$

where $v_1(t)$, $v_2(t)$ and $v_3(t)$ are the control functions. Defining error functions as

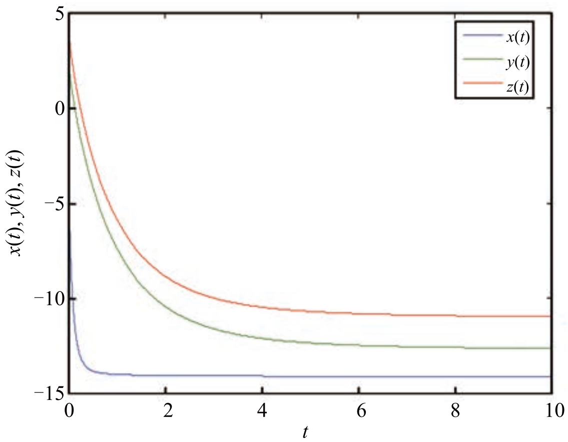
$$e_1=x_2-x_1,\quad e_2=y_2-y_1,\quad e_3=z_2-z_1.$$



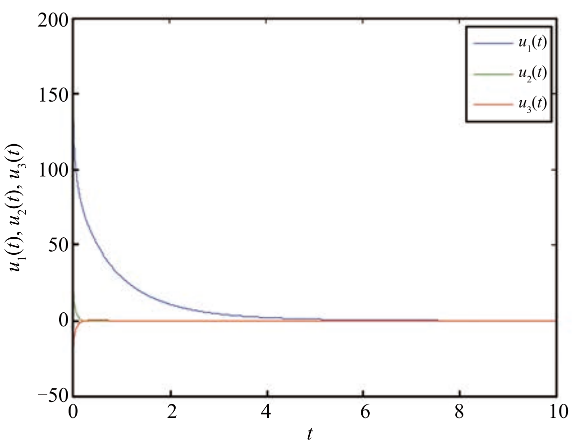
(a)



(b)



(c)



(d)

Fig. 6. Plots of $x(t)$, $y(t)$, $z(t)$of the controlled system (20). (a) At the equilibrium point P_1. (b) At the equilibrium point P_2. (c) At the equilibrium point P_3. (d) Plots of control functions $u_1(t)$, $u_2(t)$, $u_3(t)$ at P_1.

We obtain the following error system as

$$\begin{aligned}\frac{d^q e_1}{dt^q} &= \mu'(e_2 - e_3) - be_1 + (a-b)x_1 + (\mu' - \mu)y_1 \\ &\quad - \mu' z_1 + v_1(t) \\ \frac{d^q e_2}{dt^q} &= ce_1 - e_2 + (c-1)x_1 + x_2 z_2 - x_1 z_1 + v_2(t) \\ \frac{d^q e_3}{dt^q} &= -ce_1 - e_3 - x_1 c - x_2 y_2 + x_1 y_1 + v_3(t). \qquad (24)\end{aligned}$$

In order to stabilize the error system, let us consider the Lyapunov function as

$$V(e) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2\right). \qquad (25)$$

Choosing the control functions as

$$\begin{aligned}v_1(t) &= -\mu'(e_2 - e_3) - (a-b)x_1 - (\mu' - \mu)y_1 + \mu' z_1 \\ v_2(t) &= -ce_1 - (c-1)x_1 - x_2 z_2 + x_1 z_1 \\ v_3(t) &= ce_1 + x_1 c + x_2 y_2 - x_1 y_1\end{aligned}$$

the q-th order derivative of the Lyapunov function $V(e)$ becomes $\frac{d^q V(e)}{dt^q} \leq -be_1^2 - e_2^2 - e_3^2 < 0$, which concludes that $\lim_{t\to\infty} \|e(t)\| = 0$, and hence the synchronization between master and response systems is achieved.

VI. Numerical Simulation and Results

In this section, we take the earlier considered values of the parameters of systems. The initial conditions of master and slave systems are $(x_1(0),\ y_1(0),\ z_1(0)) = (0.1,\ 1.2,\ 0.5)$ and $(x_2(0),\ y_2(0),\ z_2(0)) = (-2,\ 3,\ 5)$, respectively. Hence the initial conditions of error system will be $(e_1(0),\ e_2(0),\ e_3(0)) = (-2.1,\ 1.8,\ 4.5)$. During synchronization of the systems the time step size is taken as 0.005. The synchronization between $x_1 - x_2$, $y_1 - y_2$ and $z_1 - z_2$ are depicted through Figs. 7−10 at $q = 0.7,\ 0.9,\ 0.981,\ 1.0$. The time for synchronization of the considered fractional order chaotic systems clearly exhibits that it takes less time for synchronization when the order of the derivative approaches from standard order to the fractional order.

VII. Conclusion

The authors have achieved four important goals through the analysis of the present study. First one is the stability analysis to locate the range of fractional order beyond which the systems show chaotic behavior. Second one is the synchronization between the considered fractional order systems and also chaos control of both the systems using nonlinear control method. The third one is the proper design of the control functions so that the error states decay to zero as time approaches infinity which helps to get the required time for synchronization. The most important part of the study is the comparison of time of synchronization through effective numerical simulation and graphical presentations for different particular cases as systems pair approaches from standard order to fractional order. The authors believe that the outcome of the results will be appreciated and utilized by the scientists and engineers working in the field of atmospheric science and oceanography.

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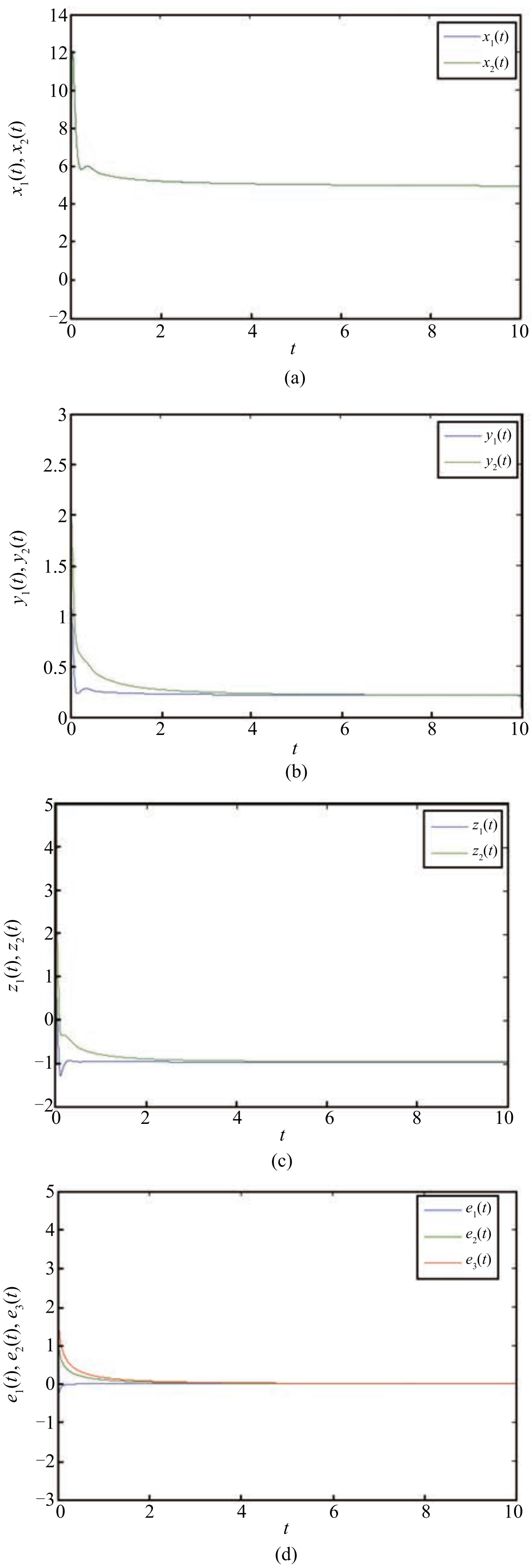

Fig. 7. State trajectories of master system (22) and slave system (23) for fractional order $q = 0.7$. (a) Synchronization between x_1 and x_2. (b) Synchronization between y_1 and y_2. (c) Synchronization between z_1 and z_2. (d) The evolution of the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$.

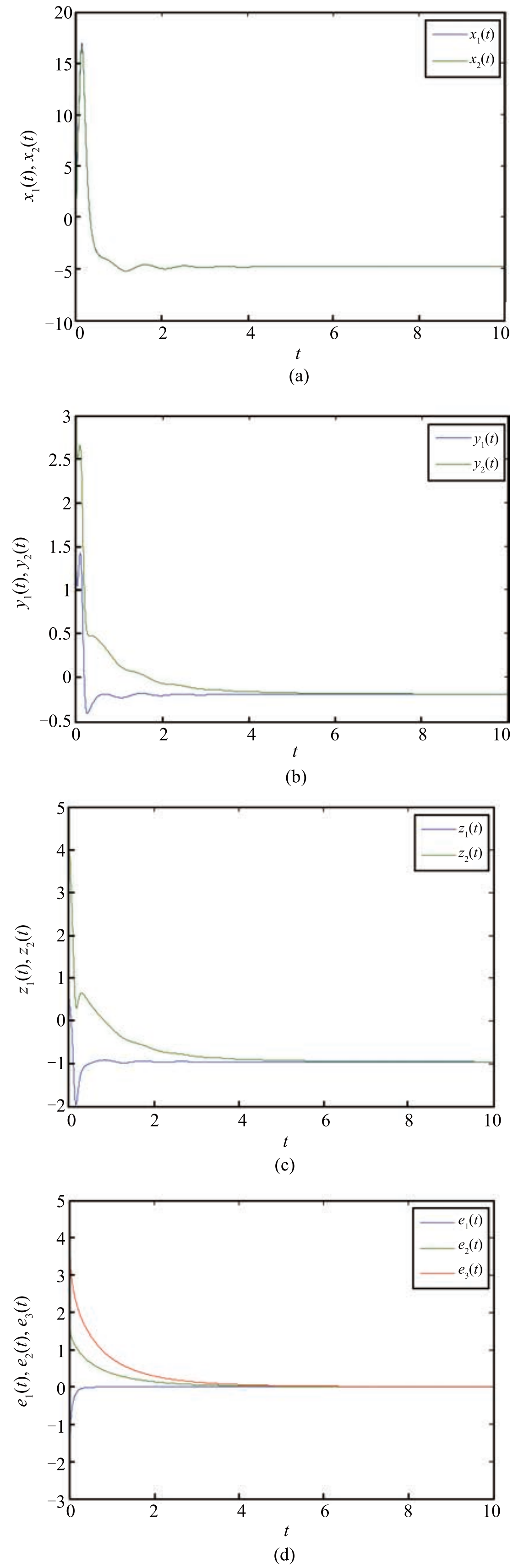

Fig. 8. State trajectories of the systems (22) and (23) for fractional order $q = 0.9$. (a) Synchronization between x_1 and x_2. (b) Synchronization between y_1 and y_2. (c) Synchronization between z_1 and z_2. (d) The evolution of the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$.

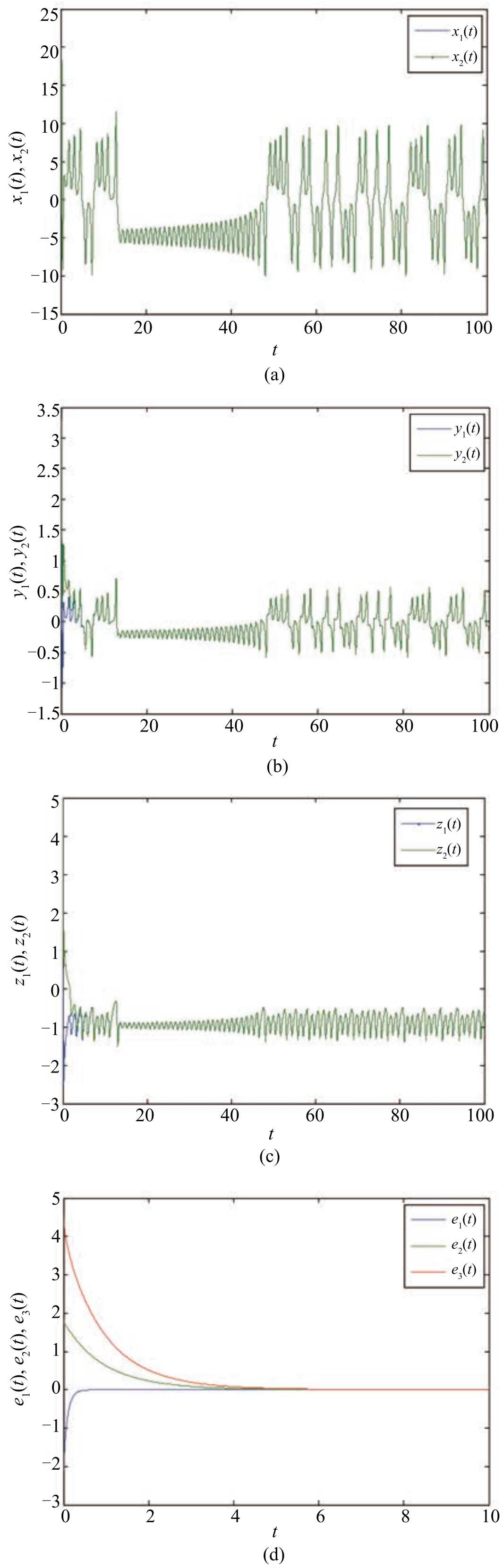

Fig. 9. State trajectories of the systems (22) and (23) for order $q = 0.981$. (a) Synchronization between x_1 and x_2. (b) Synchronization between y_1 and y_2. (c) Synchronization between z_1 and z_2. (d) The evolution of the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$.

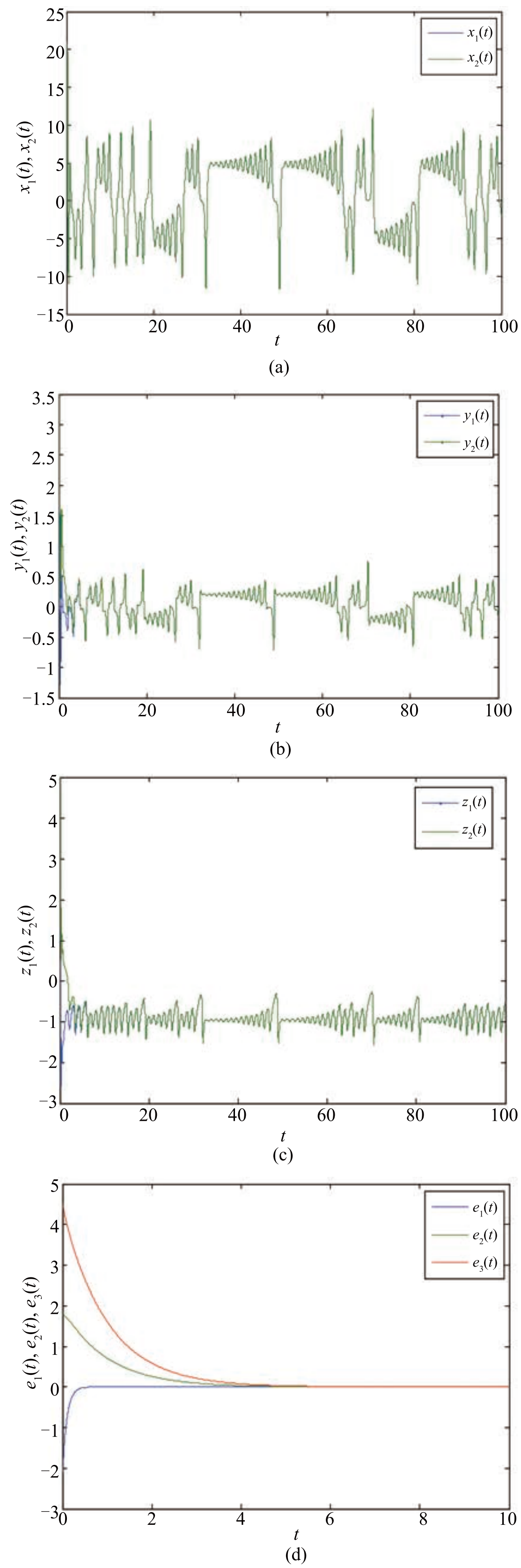

Fig. 10. State trajectories of the systems (22) and (23) for $q = 1$. (a) Synchronization between x_1 and x_2. (b) Synchronization between y_1 and y_2. (c) Synchronization between z_1 and z_2. (d) Evolution of the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$.

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