Numerical Solutions of Fractional Differential Equations by Using Fractional Taylor Basis

Vidhya Saraswathy Krishnasamy, Somayeh Mashayekhi, and Mohsen Razzaghi

Abstract—In this paper, a new numerical method for solving fractional differential equations (FDEs) is presented. The method is based upon the fractional Taylor basis approximations. The operational matrix of the fractional integration for the fractional Taylor basis is introduced. This matrix is then utilized to reduce the solution of the fractional differential equations to a system of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of this technique.

Index Terms—Caputo derivative, fractional differential equations (FDEs), fractional Taylor basis, operational matrix, Riemann-Liouville fractional integral operator.

I. INTRODUCTION

THE fractional differential equations (FDEs) have drawn increasing attention and interest due to their important applications in science and engineering. A history of the development of fractional differential operators can be found in [1]—[3].

Many mathematical modelings contain FDEs. To mention a few, fractional derivatives are used in visco-elastic systems [4], economics [5], continuum and statistical mechanics [6], solid mechanics [7], electrochemistry [8], biology [9] and acoustics [10]. Generally speaking, most of the FDEs do not have exact analytic solutions. Therefore, seeking numerical solutions of these equations becomes more and more important. Recently, several numerical methods to solve FDEs have been given, such as Fourier transforms [11], Laplace transforms [12], Adomian decomposition method [13], variational iteration method [14], the power series method [15], truncated fractional power series method [16], fractional differential transform method (FDTM) [17], homotopy analysis method [18], fractional-order Legendre functions method [19], modified homotopy perturbation method (MHPM) [20] and enhanced homotopy perturbation method (EHPM) [21].

Moreover, for solving FDEs in [22], the Bernstein polynomials are used to solve the fractional Riccati type differential equations. In [22], the Bernstein polynomials were first expanded into fractional Taylor polynomials. The operational matrix of fractional differentiation (OMFD) of fractional Taylor polynomials were then used for calculating OMFD for Bernstein polynomials. In addition, the Chebyshev, Legendre and Bernoulli wavelets operational matrices of fractional integration (OMFI) were calculated in [23]—[25], respectively. For obtaining OMFI in [23], [24], these wavelets were first expanded into block-pulse functions. Then, OMFI of block-pulse were used for calculating OMFI for Chebyshev and Legendre wavelets in [23], [24], respectively. In [25], for obtaining the OMFI for Bernoulli wavelets, these wavelets were expanded into Bernoulli polynomials.

In this paper, a new numerical method for solving the initial and boundary value problems for fractional differential equations is presented. The method is based upon the fractional Taylor basis approximations. The OMFI for the fractional Taylor basis is calculated. This matrix is then utilized to reduce the solution of the FDEs to the solution of algebraic equations. This method is applicable for linear equations or nonlinear equations with square nonlinearities.

The outline of this paper is as follows: In Section II, we introduce some necessary definitions and properties of fractional calculus. Section III is devoted to the basic formulation of the fractional Taylor basis. In Section IV, we derive the Fractional Taylor OMFI. In Section V, the problem statement is given. Section VI is devoted to the numerical method for solving the initial and boundary value problems for FDEs and, in Section VII we report our numerical findings and demonstrate the accuracy of the proposed numerical scheme by considering five numerical examples.

II. PRELIMINARIES

A. The Fractional Integral and Derivative

In this section, we present some notations, definitions, and preliminary facts of the fractional calculus theory which will be used further in this work.

Definition 1: The Riemann-Liouville fractional integral operator of order \( \alpha \) is defined as [12]

\[
I^\alpha y(t) = \int_{0}^{t} \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} y(s)ds, \quad \alpha > 0
\]

\[
y(t), \quad \alpha = 0.
\]

The Riemann-Liouville fractional integral operator has the following properties:

\[
I^\alpha t^\gamma = \frac{t^{\gamma+\alpha}}{\Gamma(\gamma+\alpha+1)}, \quad \alpha \geq 0; \gamma > -1
\]

\[
I^\alpha I^\beta y(t) = I^{\alpha+\beta} y(t) = I^{\alpha+\beta} y(t), \quad \alpha, \beta > 0.
\]
Also the fractional integral is a linear operator, that is for constants \( \lambda_1 \) and \( \lambda_2 \), we have
\[
I^\alpha(\lambda_1y_1(t) + \lambda_2y_2(t)) = \lambda_1I^\alpha y_1(t) + \lambda_2I^\alpha y_2(t).
\]

Definition 2: The Caputo fractional derivative of order \( \alpha \) is defined as [12]
\[
D^\alpha y(t) = I^{n-\alpha} \left( \frac{d^n}{dt^n} y(t) \right), \quad n-1 < \alpha \leq n; \quad n \in \mathbb{N}. \tag{3}
\]

The fractional integral operator and fractional derivative operator do not commute in general, but we have the following property
\[
I^\alpha (D^\alpha y(t)) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} \frac{t^k}{k!}. \tag{4}
\]

III. THE PROPERTIES OF FRACTIONAL TAYLOR BASIS

A. Fractional Taylor Basis Vector

In this paper, we define the fractional Taylor basis vector as
\[
T_{m\gamma}(t) = [1, t^\gamma, t^{2\gamma}, \ldots, t^{m\gamma}]^T \tag{5}
\]
where \( m \) is a positive integer and \( \gamma > 0 \), is a real number.

B. Function Approximation

Let \( H = L^2[0,1] \), and assume that \( T_{m\gamma} \subset H \), \( S = \{1, t^\gamma, t^{2\gamma}, \ldots, t^{m\gamma}\} \) and \( y \) be an arbitrary element in \( H \). Since \( S \) is a finite dimensional vector subspace of \( H \), \( y \) has a unique best approximation out of \( S \) such as \( y_0 \in S \), that is
\[
\forall \tilde{y} \in S, \quad \|y - y_0\| \leq \|y - \tilde{y}\|.
\]

Since \( y_0 \in S \), there exist unique coefficients \( c_0, c_1, c_2, \ldots, c_m \), such that
\[
y \approx y_0 = \sum_{i=0}^{m} c_i t^{i\gamma} = C^T T_{m\gamma}(t) \tag{6}
\]
where
\[
C^T = [c_0, c_1, c_2, \ldots, c_m]. \tag{7}
\]

C. Error Bound for the Best Approximation

To obtain the error bound for the best approximation, we use the following formula.

Generalized Taylor formula [15]: Suppose that \( D^{k\gamma}y(t) \in C[0,1] \) for \( k = 0, 1, \ldots, m \), where \( 0 < \gamma \leq 1 \), then
\[
y(t) = \sum_{i=0}^{m} \frac{(t)^i}{\Gamma(i+1)} [D^{i\gamma} y(t)]_{t=0} + R_{m\gamma}(t, 0) \tag{8}
\]
where \( D^{\gamma} = D^{\gamma_1} D^{\gamma_2} \cdots D^{\gamma_t} \), with \( D^{\gamma} \) defined similar to \( D^\alpha \) in (3), and
\[
R_{m\gamma}(t, 0) = \frac{\Gamma((m+1)\gamma + 1)}{\Gamma(m+1\gamma + 1)} [D^{(m+1)\gamma} y(t)]_{t=0} \quad 0 \leq \xi \leq t; \quad \forall t \in [0,1].
\]

Theorem 1: Let \( y_0 \) be the best approximation of \( y \) out of \( S \) and suppose \( D^{k\gamma}y(t) \in C[0,1], k = 0, 1, \ldots, \), then
\[
\|y(t) - y_0(t)\| \leq \frac{M_\gamma}{\Gamma((m+1)\gamma + 1)} \sqrt{\frac{1}{2(m+1)\gamma + 1}} \tag{9}
\]
where
\[
M_\gamma = \sup_{t \in [0,1]} |D^{(m+1)\gamma} y(t)|.
\]

Proof: Similar to [19], since \( y_0 \) is the best approximation of \( y \) out of \( S \), by using (8) we have
\[
\begin{align*}
\|y - y_0\|_2 & \leq \frac{M_\gamma^2}{\Gamma((m+1)\gamma + 1)^2} \int_0^1 (t)^{2(m+1)\gamma} dt \\
& = \frac{M_\gamma^2}{\Gamma((m+1)\gamma + 1)^2} \frac{1}{2(m+1)\gamma + 1}. \tag{10}
\end{align*}
\]

By using (9), the result can be obtained.

D. Error Bound for Fractional Integration

In this section we obtain the error bound for \( I^\alpha y(t) \).

Theorem 2: Suppose all the conditions in Theorem 1 are true and \( \alpha > 1 \), then
\[
\|I^\alpha y(t) - I^\alpha y_0(t)\| \leq \frac{M_\gamma}{\Gamma(m+1)\gamma} \Gamma(m+1) \Gamma(\alpha) \sqrt{\frac{1}{2(m+1)\gamma + 1}}. \tag{11}
\]

Proof: By using Definition 1, we have
\[
\begin{align*}
\|I^\alpha y(t) - I^\alpha y_0(t)\|_2 & \leq \frac{1}{\Gamma(\alpha)} \left[ \int_0^1 \|(t-s)^{\alpha-1}(y(s) - y_0(s))\|_2 ds \right. \\
& \leq \frac{1}{\Gamma(\alpha)} \left. \int_0^1 \||1-s)^{\alpha-1}(y(s) - y_0(s))\|_2 ds \right. \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^1 \|\tilde{y}(s) - y_0(s)\|_2 ds. \tag{12}
\end{align*}
\]

By using (9) and (12), the result can be obtained.

IV. THE OPERATIONAL MATRICES

A. Operational Matrix of the Fractional Integration

In this section we derive the fractional Taylor operational matrix of the fractional integration.

By using (1) and (5), we have
\[
I^\alpha(T_{m\gamma}(t)) = \begin{bmatrix}
\frac{1}{\Gamma(\alpha + 1)} & \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} & t^{\gamma + \alpha} \\
\frac{\Gamma(2\gamma + 1)}{\Gamma(2\gamma + \alpha + 1)} & \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} & \ldots & \frac{\Gamma(m\gamma + 1)}{\Gamma(m\gamma + \alpha + 1)} & t^{m\gamma + \alpha} \\
\end{bmatrix}^T \tag{13}
\]
where
\[
F_\alpha = \text{diag} \left[ \frac{1}{\Gamma(\alpha + 1)}, \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)}, \ldots, \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} \right].
\]

Equation (11) can be rewritten as
\[
I^\alpha(T_{m\gamma}(t)) = t^\alpha G_\alpha \ast T_{m\gamma}(t) \tag{14}
\]
where
where \( S \) Taylor vectors will also be used.

and * denotes term by term multiplication of two matrices of the same dimensions.

**B. Operational Matrix of Product**

The following property of the product of two fractional Taylor vectors will also be used.

\[
I^\alpha (T_{m\gamma}(t)T^T_{m\gamma}(t)) = t^\alpha S_\alpha \ast (T_{m\gamma}(t)T^T_{m\gamma}(t))
\]

where \( S_\alpha \) is given by

\[
S_\alpha = \begin{bmatrix}
\frac{1}{\Gamma(\alpha + 1)} & \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} & \ldots & \frac{\Gamma(m\gamma + 1)}{\Gamma(m\gamma + \alpha + 1)} \\
\frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} & \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} & \ldots & \frac{\Gamma((m+1)\gamma + 1)}{\Gamma((m+1)\gamma + \alpha + 1)} \\
\frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} & \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} & \ldots & \frac{\Gamma((m+2)\gamma + 1)}{\Gamma((m+2)\gamma + \alpha + 1)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\Gamma((m\gamma + 1)}{\Gamma((m\gamma + \alpha + 1)} & \frac{\Gamma((m+1)\gamma + 1)}{\Gamma((m+1)\gamma + \alpha + 1)} & \ldots & \frac{\Gamma(2m\gamma + 1)}{\Gamma(2m\gamma + \alpha + 1)}
\end{bmatrix}
\]

To illustrate the calculation procedure, by using (5), we have

\[
T_{m\gamma}(t)T^T_{m\gamma}(t) = \begin{bmatrix}
1 & t^\gamma & t^{2\gamma} & \ldots & t^{m\gamma} \\
t^\gamma & t^{2\gamma} & t^{3\gamma} & \ldots & t^{(m+1)\gamma} \\
t^{2\gamma} & t^{3\gamma} & t^{4\gamma} & \ldots & t^{(m+2)\gamma} \\
\vdots & \vdots & \ddots & \vdots \\
t^{m\gamma} & t^{(m+1)\gamma} & t^{(m+2)\gamma} & \ldots & t^{2m\gamma}
\end{bmatrix}
\]

From (1) and (15), we get (16), shown at the bottom of the page.

Therefore from (13) and (15), we get \( S_\alpha \) in (14).

### V. Problem Statement

In this paper we focus on the following FDE problems [24].

**A. Problem a**

Caputo fractional differential equation

\[
D^\alpha y(t) = f(t, y(t), D^\beta y(t))
\]

\[0 \leq t \leq 1; \quad 0 < \alpha \leq 2; \quad 0 \leq \beta \leq \alpha\]

with the initial conditions

\[y(0) = Y_0, \quad y'(0) = Y_1.\]

The existence and uniqueness results for solution of this problem are given in [26].

**B. Problem b**

Caputo fractional differential equation in (17) with the boundary conditions

\[y(0) = Y_0, \quad y(1) = Y_1.\]

For this problem, we have the following Lemma 1.

**Lemma 1:** Assume that \( f : [0, 1] \times \mathbb{R}^\times \mathbb{R} \) is continuous.

Then \( y(t) \in \mathbb{C}[0, 1] \) is a solution of the boundary value problem in (17) and (19) if and only if \( y(t) \) is the solution of [24].

\[y(t) = I^\alpha f(t, y(t), D^\beta y(t)) - tI^\alpha f(1, y(1), D^\beta y(1)) + Y_1 - Y_0.\]

The existence and uniqueness results for solution of this problem are given in [24].

### VI. The Numerical Method

In this section, we use the fractional Taylor vector in (5) for solving Problem a given in (17) and (18) and Problem b given in (17) and (19).

**A. Problem a**

In this case, by using (4) and (17), we have

\[y(t) - \sum_{k=0}^{n-1} y^k(t) \frac{t^k}{k!} = I^\alpha f(t, y(t), D^\beta y(t)).\]
Substituting (6) and (18) in (21), we obtain
\[ C^T T_{m\gamma}(t) - Y_0 - Y_1 t = I^\alpha f(t, C^T T_{m\gamma}(t), D^\beta (C^T T_{m\gamma}(t))). \]  
(22)

Next, we use the operational matrices obtained in Section 4 as needed and collocate (22) at the following equidistant nodes \( t_i \) given by
\[ t_i = \frac{i}{m}, \quad i = 0, 1, 2, \ldots, m. \]  
(23)

These equations give \( m + 1 \) algebraic equations, which can be solved for the unknown vector \( C^T \) using Newton’s iterative method. It is known that the initial guess for Newton’s iterative method is very important. According to the conditions in (18) the solution \( y(t) \) will pass through the point \((0, Y_0)\) and have a slope \( Y_1 \) at this point. We choose our initial guess \( y_0(t) \) such that \( y_0(t) = Y_1 t + Y_0 \).

B. Problem b

For Problem b, by substituting (6) in (20) we get (24), shown at the bottom of the page.

By using the operational matrices obtained in Section IV wherever needed and collocating (24) at the equidistant nodes \( t_i \), given in (23), we get a system of algebraic equations, which can be solved for the unknown vector \( C^T \) using Newton’s iterative method. In this case, the initial values required to start Newton’s iterative method have been chosen by taking \( y(t) \) as a linear function between the initial value \( y(0) = Y_0 \) and the final value \( y(1) = Y_1 \).

VII. ILLUSTRATIVE EXAMPLES

In this section, five examples are given to demonstrate the applicability and accuracy of our method. Examples 1–4 are initial value problems and Example 5 is a boundary value problem. Example 1 is an initial value FDE, which was first considered in [19]. The exact solution of Example 1 is a polynomial, and the exact solution can be obtained using the proposed method. Examples 2 and 3 are FDEs describing the fractional Riccati equation, which were first considered in [20] by using modified homotopy perturbation method, it was also studied in [21] by applying the enhanced homotopy perturbation method, in [22] by using Bernstein polynomials and in [25] by applying Bernoulli wavelets. For Examples 2 and 3, we compare our findings with the numerical results in [20]–[22], [25]. Example 4 was first considered in [27] by using a predictor corrector approach; it was also solved in [28] by converting the FDE to a Volterra type integral equation and in [24] by using Legendre wavelet method. For Example 4 we compare our method with [24] which has been shown to be comparable or superior to [27], [28]. Example 5 was solved in [24] by using Legendre wavelet. For Example 5, we compare our results with [24]. In Examples 2–5 the package of Mathematica ver. 9.0 has been used to solve the test problems. Here, we first give a method for selecting \( \gamma \) in (5) for our examples. We select \( \gamma = 1 \) if \( \alpha = 1 \) or \( \alpha = 2 \). Otherwise, we select \( \gamma = \alpha \). For Example 5, similar to [29] we have also used \( \gamma = \alpha - \lfloor \alpha \rfloor \), and we get better results than \( \alpha \). Here \( \lfloor \alpha \rfloor \) is the floor function which is the greatest integer less than or equal to the \( \alpha \).

A. Example 1

Consider the following linear fractional differential equation given in [19].
\[ D^2 y(t) + D^\gamma y(t) + y(t) = 1 + t \]
\[ 0 < t \leq 1; \quad y(0) = 1; \quad y'(0) = 1. \]  
(25)

The exact solution of this problem is
\[ y(t) = 1 + t. \]

Here, we solve this problem by using the proposed method with \( \gamma = 1 \) and \( m = 1 \).

Let
\[ y(t) \approx C^T T_{m\gamma}(t) = [c_0, c_1]\begin{pmatrix} 1 \\ t \end{pmatrix}. \]  
(26)

By using (1)–(4) and (25), we have
\[ y(t) - 1 - t + I^{1/2} \left( I^{1/2} D^{1/2} y(t) \right) + I^2 y(t) = \frac{t^2}{2} + \frac{t^3}{6}. \]  
(27)

By substituting (26) in (27), we get
\[ C^T T_{m\gamma}(t) - 1 - t + I^{1/2} \left( C^T T_{m\gamma}(t) - y(0) - y'(0) t \right) + I^2 C^T T_{m\gamma}(t) = \frac{t^2}{2} + \frac{t^3}{6}. \]

From (12), we have (28), shown at the bottom of the page. where
\[ G_1 = \left[ \begin{array}{c} \frac{1}{\Gamma(\frac{3}{2})} \\ \frac{1}{\Gamma(\frac{5}{2})} \end{array} \right], \quad G_2 = \left[ \begin{array}{c} \frac{1}{\Gamma(\frac{3}{2})} \\ \frac{1}{\Gamma(\frac{5}{2})} \end{array} \right]. \]  
(29)

Substituting (29) in (28) and collocating the resulting equation at \( t_0 = 0 \) and \( t_1 = 1 \), we get
\[ c_0 = 1, \quad c_1 = 1. \]

Then, by using (26), we get \( y(t) = 1 + t \), which is the exact solution.

B. Example 2

Consider the fractional Riccati differential equation [22].
\[ D^\alpha y(t) + y^2(t) = 1, \quad y(0) = 0; \quad 0 < \alpha \leq 1. \]  
(30)

\[ C^T T_{m\gamma}(t) - I^\alpha f(t, C^T T_{m\gamma}(t), D^\beta (C^T T_{m\gamma}(t))) + t I^\alpha \left( f(1, C^T T_{m\gamma}(t), D^\beta (C^T T_{m\gamma}(t))) \right) - (Y_1 - Y_0) t - Y_0 = 0 \]  
(24)

\[ C^T T_{m\gamma}(t) - 1 - t + I^{1/2} C^T \left( G_1 \ast T_{m\gamma}(t) - \frac{t^2}{\Gamma(\frac{3}{2})} - \frac{t^2}{\Gamma(\frac{5}{2})} \right) + I^2 C^T (G_2 \ast T_{m\gamma}(t)) = \frac{t^2}{2} + \frac{t^3}{6} \]  
(28)
where 

\[ C \quad \text{and} \quad \gamma \]

with \( m \) and \( \alpha \)

\[ T - 0.6 \quad 4.0657 \times 10^{-0.4} \quad 2.5969 \times 10^{-0.4} \]

\[ -0.2 \quad 5.1734 \times 10^{-2} \quad 0 \]

\[ t \]

(22), (30), and (31), we get

\[ \text{Proposed method,} \]

\[ \text{BPM [22],} \]

\[ \text{Proposed method,} \]

\[ \text{BPM [22],} \]

\[ \text{Proposed method,} \]

\[ \text{IABMM [21],} \]

\[ \text{EHPM [21],} \]

\[ \text{MHPM [20],} \]

\[ \text{Proposed method,} \]

\[ \text{BPM [22],} \]

\[ \text{Proposed method,} \]

\[ \text{BPM [22],} \]

\[ \text{Proposed method,} \]

\[ \text{IABMM [21],} \]

\[ \text{EHPM [21],} \]

\[ \text{MHPM [20],} \]

\[ \text{Proposed method,} \]

\[ \text{BPM [22],} \]

\[ \text{Proposed method,} \]

\[ \text{BPM [22],} \]

\[ \text{Proposed method,} \]

\[ \text{IABMM [21],} \]

\[ \text{EHPM [21],} \]

\[ \text{MHPM [20],} \]

\[ \text{Comparison of Numerical Results for } \alpha = 0.75 \]

\[ \text{Comparison of Numerical Results for } \alpha = 0.9 \]

\[ \text{Comparison of Absolute Error for } \alpha = 1 \]

The exact solution of this problem for \( \alpha = 1 \) is

\[ y(t) = \frac{e^{t^2} - 1}{e^{t^2} + 1}. \]

To compare the proposed method with [20]–[22], we solve (30) for \( \alpha = 0.75, \alpha = 0.9, \) and \( \alpha = 1. \) Now, we solve (30), with \( m = 3 \) and \( \gamma = 0.75. \) Let

\[ y(t) \equiv C^T T_{m\gamma}(t) \]

(31)

where

\[ C^T = [c_0, c_1, c_2, c_3] \]

and

\[ T_{m\gamma}(t) = [1, t^{0.75}, t^{1.5}, t^{2.25}]^T. \]

By using (22), (30), and (31), we get

\[ C^T T_{m\gamma}(t) + t^\alpha C^T (S_\alpha (T_{m\gamma}(t) T_{m\gamma}(t)^T)) C - \frac{t^\alpha}{\Gamma(\alpha + 1)} = 0 \]

(32)

where

\[ T_{m\gamma}(t) T_{m\gamma}(t)^T = \begin{bmatrix} 1 & t^{0.75} & t^{1.5} & t^{2.25} \\ t^{0.75} & 1 & t^{1.5} & t^{2.25} \\ t^{1.5} & t^{2.25} & t^2 & t^{3} \\ t^{2.25} & t^3 & t^{3.75} & t^{4.5} \end{bmatrix} \]

and

\[ S_\alpha(t) = \begin{bmatrix} 1 \times 10^8 & 0.00015 & 0.00015 & 0.00015 \\ 0.00015 & 0.00015 & 0.00015 & 0.00015 \\ 0.00015 & 0.00015 & 0.00015 & 0.00015 \\ 0.00015 & 0.00015 & 0.00015 & 0.00015 \end{bmatrix}. \]

By collocating (32) at the nodes given in (23), and solving the resulting equations we get,

\[ c_0 = 0, \quad c_1 = 1.03094, \quad c_2 = -0.165321, \quad \text{and} \quad c_3 = -0.1044. \]

Then, by using (31), we have

\[ y(t) = 1.03094 t^{0.75} - 0.165321 t^{1.5} - 0.1044 t^{2.25}. \]

In Tables I and II, we compare our results with the solutions of the modified homotopy perturbation method (MHPM) in [20], the improved Adams-Bashforth-Moulton method (IABMM) in [21], the enhanced homotopy perturbation method (EHPM) in [21] and with the Bernstein polynomials method (BPM) in [22] for different values of \( m \). In Table III, we compare the absolute error of our method for \( \gamma = \alpha = 1 \) with MHPM [20] and BPM [22], for different values of \( m \). In Tables I–III, \( N \) represents the degree of the Bernstein polynomial used in [22]. Also, Fig.1 shows the approximate solutions obtained for different values of \( \alpha \) using the proposed method with \( m = 5 \).
these results, it is seen that the approximate solutions converge to the exact solution for \( \alpha = 1 \). In addition, the absolute difference between the exact and approximate solutions for \( \alpha = 1 \) with \( m = 5 \) is plotted in Fig. 2. The absolute difference between the exact and approximate solutions for \( k = 1 \) and \( M = 5 \) or \( \hat{m} = 2^{k-1}M = 5 \) and \( \alpha = 1 \) by Bernoulli wavelets method is plotted in [25]. Here, \( k \) and \( M \) are the order of wavelets and Bernoulli polynomials respectively. From our figures and those in [25], we can conclude that the result obtained by the proposed method has less error compared to Bernoulli wavelets method.

Fig. 1. Comparison of the computed solutions for different values of \( \alpha \) with exact solution for \( \gamma = \alpha = 1 \) for Example 2 with \( m = 5 \).

Fig. 2. The absolute error for \( \gamma = \alpha = 1 \) for Example 2 with \( m = 5 \).

C. Example 3

Consider the following Riccati fractional differential equation given in [22].

\[ D^\alpha y(t) = 2y(t) - y^2(t) + 1, \quad y(0) = 0; \quad 0 < \alpha \leq 1. \tag{33} \]

To solve this problem by using the proposed method, we let

\[ y(t) \cong C^T T_{m\gamma}(t) \tag{34} \]

where \( C^T \) and \( T_{m\gamma}(t) \) are given in (5) and (7) respectively. Using (22), (33), and (34), we have

\[
C^T T_{m\gamma}(t) - 2t^\alpha C^T (G_\alpha * T_{m\gamma}(t)) + t^\alpha C^T (S_\alpha * (T_{m\gamma}(t)T_{m\gamma}^T(t)))C - \frac{t^\alpha}{\Gamma(\alpha+1)} = 0. \tag{35}
\]

Now, by collocating (35) at the nodes given in (23), we get \( m + 1 \) nonlinear algebraic equations which can be solved for the unknown vector \( C^T \) using Newton’s iterative method.

It is well known that the initial guesses for Newton’s iterative method are very important. For this problem, by using \( y(0) = 0 \), and (34), we choose the initial guesses such that

\[ C^T T_{m\gamma}(0) = 0. \]

The exact solution of this problem for \( \alpha = 1 \) is

\[ y(t) = 1 + \sqrt{2} \tanh \left( \sqrt{2} t + \frac{1}{2} \ln \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right). \]

Table IV shows the comparison of our numerical results with [20]–[22] for \( \gamma = \alpha = 0.9 \). In Table V, we compare the absolute error of our numerical method with [20] and [22] for \( \alpha = 1 \). Also, Fig. 3 shows the approximate solutions obtained for different values of \( \alpha \) using the proposed method with \( m = 5 \). From these results, it is seen that the approximate solutions converge to the exact solution for \( \alpha = 1 \). From Table V it is seen that our results with \( m = 18 \) has less error than the results in the table given in [25], with \( k = 2 \) and \( M = 10 \) or \( \hat{m} = 2^{k-1}M = 20 \) using Bernoulli wavelets method.

Fig. 3. Comparison of the computed solutions for different values of \( \alpha \) with exact solution for \( \alpha = 1 \) for Example 3 with \( m = 5 \).

D. Example 4

Consider the FDE [24]

\[ D^\alpha y(t) + y(t) = 0, \quad 0 < \alpha \leq 2 \tag{36} \]

with \( y(0) = 1 \) and \( y'(0) = 0 \). The condition \( y'(0) = 0 \) is for \( 1 < \alpha \leq 2 \) only.

The exact solution of this problem is

\[ y(t) = E_\alpha(-t^\alpha) \tag{24}, \]

where

\[ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \]

is the Mittag-Leffler function with order \( \alpha \). To solve this problem by using the proposed method, similar to (34) in Example 3 we let

\[ y(t) \cong C^T T_{m\gamma}(t). \tag{37} \]

Using (22), (36), and (37), we have

\[ C^T T_{m\gamma}(t) - 1 + t^\alpha C^T (G_\alpha * T_{m\gamma}(t)) = 0. \]
By collocating at the points given in (23) we get \( m + 1 \) algebraic equations, which can be solved for the unknown vector \( C^T \). Table VI shows the absolute error obtained for different values of \( t \) and for \( \alpha = 1.5 \) by using the proposed method with different values of \( m \) and the Legendre wavelets method (LWM) in [24], with \( k = 8 \) and \( M_1 = 3 \) or \( \tilde{M} = 2^{k-1} M_1 = 384 \). Here, \( M_1 \) shows the order of Legendre polynomials. In Table VII, the absolute error obtained using the proposed method for different values of \( \alpha \) with \( m = 10 \) is given.
TABLE IX

<table>
<thead>
<tr>
<th>$t_i$</th>
<th>$\alpha = 1.1$</th>
<th>$\alpha = 1.3$</th>
<th>$\alpha = 1.5$</th>
<th>$\alpha = 1.6$</th>
<th>$\alpha = 1.8$</th>
<th>$\alpha = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.33357E−17</td>
<td>5.01444E−18</td>
<td>2.71728E−18</td>
<td>7.78999E−18</td>
<td>5.36765E−18</td>
<td>8.4514E−19</td>
</tr>
<tr>
<td>0.2</td>
<td>1.04083E−17</td>
<td>6.07153E−18</td>
<td>1.22515E−17</td>
<td>1.85399E−17</td>
<td>1.55041E−17</td>
<td>2.19551E−18</td>
</tr>
<tr>
<td>0.3</td>
<td>6.93889E−18</td>
<td>0</td>
<td>3.1225E−17</td>
<td>2.38524E−17</td>
<td>2.32019E−17</td>
<td>2.1684E−19</td>
</tr>
<tr>
<td>0.4</td>
<td>2.77556E−17</td>
<td>1.04083E−17</td>
<td>5.89086E−16</td>
<td>1.9082E−17</td>
<td>2.34188E−17</td>
<td>1.04083E−17</td>
</tr>
<tr>
<td>0.5</td>
<td>5.55112E−17</td>
<td>1.38778E−16</td>
<td>9.71445E−17</td>
<td>2.7756E−17</td>
<td>2.77556E−17</td>
<td>9.71445E−17</td>
</tr>
<tr>
<td>0.6</td>
<td>5.55112E−17</td>
<td>0</td>
<td>2.77556E−17</td>
<td>2.77556E−17</td>
<td>2.77556E−17</td>
<td>2.77556E−17</td>
</tr>
<tr>
<td>0.7</td>
<td>0</td>
<td>5.55112E−17</td>
<td>1.66533E−16</td>
<td>1.66533E−16</td>
<td>1.66533E−16</td>
<td>1.66533E−16</td>
</tr>
<tr>
<td>0.8</td>
<td>1.11022E−16</td>
<td>1.11022E−16</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3.33067E−16</td>
<td>3.33067E−16</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

### TABLE VIII

<table>
<thead>
<tr>
<th>$t_i$</th>
<th>LWM[24]</th>
<th>Proposed method</th>
<th>Proposed method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = 12$</td>
<td>$\gamma = \alpha$</td>
<td>$\gamma = \alpha - \lfloor \alpha \rfloor$</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>9.6996E−5</td>
<td>1.3443E−1</td>
<td>1.0679E−17</td>
</tr>
<tr>
<td>0.2</td>
<td>9.3927E−4</td>
<td>2.6884E−9</td>
<td>1.3877E−17</td>
</tr>
<tr>
<td>0.3</td>
<td>1.5087E−3</td>
<td>4.0331E−9</td>
<td>8.6732E−18</td>
</tr>
<tr>
<td>0.4</td>
<td>3.3989E−4</td>
<td>5.3764E−9</td>
<td>4.1633E−17</td>
</tr>
<tr>
<td>0.5</td>
<td>2.4160E−3</td>
<td>6.7323E−9</td>
<td>2.7755E−17</td>
</tr>
<tr>
<td>0.6</td>
<td>3.1023E−4</td>
<td>8.0597E−9</td>
<td>2.7755E−17</td>
</tr>
<tr>
<td>0.7</td>
<td>1.4799E−3</td>
<td>9.4327E−9</td>
<td>0</td>
</tr>
<tr>
<td>0.8</td>
<td>6.3407E−4</td>
<td>1.0630E−8</td>
<td>1.6653E−16</td>
</tr>
<tr>
<td>0.9</td>
<td>4.6701E−3</td>
<td>1.3054E−8</td>
<td>2.2204E−16</td>
</tr>
</tbody>
</table>

### VIII. Conclusion

In the present work the fractional Taylor basis is used to solve FDEs. The integral operational matrix $F_\alpha$ and $S_\alpha$ have been derived. The error bounds are also included. The problem has been reduced to a problem of solving a system of algebraic equations. Illustrative examples are solved by using the proposed method to show that this approach can solve the problem effectively.

### References


Vidhya Saraswathy Krishnasamy received her undergraduate and master’s degree in mathematics from Sri G.V.G. Vishalakshi College for Women, Udumalpet, India. She graduated from Mississippi State University (MSU), USA with her Ph.D. degree in August, 2016.

She has diverse research interests, which include fractional calculus, orthogonal functions and its applications to dynamical systems, graph theory, number theory, cryptography and its applications.

Somayeh Mashayekhi graduated from Mississippi State University (MSU), USA, in 2015. She received her first Ph.D. degree from Alzahra University in 2013, and her second Ph.D. degree from MSU in 2015. She is, currently, a post doctoral research associate in computational science and engineering, Department of Mathematics, Florida State University.

Her research interests include optimal control, delay system, fractional calculus, orthogonal functions and its applications to dynamical systems.

Mohsen Razzaghi received his undergraduate degree in mathematics from the University of Sussex in England. He went to Canada and received his master’s degree in applied mathematics from the University of Waterloo. He returned to the University of Sussex and obtained his Ph.D. After that, he has taught and served in several administrative positions in Iran and in the USA. Since 1986, he has been at the Department of Mathematics and Statistics at Mississippi State University, where he is currently a professor and the department head. During the academic years 2011-2012 and 2015-2016, he was a Fulbright Scholar at the Department of Mathematics and Computer Science at the Technical University of Civil Engineering in Bucharest, Romania.

His area of research centers on optimal control, orthogonal functions and wavelets in dynamical system, and fractional calculus. He has over 150 refereed journal publications in mathematics, mathematical physics, and engineering. Corresponding author of this paper.