Criteria for Response Monotonicity Preserving in Approximation of Fractional Order Systems

Mohammad Saleh Tavazoei, *Member, IEEE*

***Abstract*—In approximation of fractional order systems, a significant objective is to preserve the important properties of the original system. The monotonicity of time/frequency responses is one of these properties whose preservation is of great importance in approximation process. Considering this importance, the issues of monotonicity preservation of the step response and monotonicity preservation of the magnitude-frequency response are independently investigated in this paper. In these investigations, some conditions on approximating filters of fractional operators are found to guarantee the preservation of step/magnitude-frequency response monotonicity in approximation process. These conditions are also simplified in some special cases. In addition, numerical simulation results are presented to show the usefulness of the obtained conditions.**

***Index Terms*—Fractional order system, approximation, step response, magnitude-frequency response, monotonicity**

I. Introduction

THESE days, fractional calculus[1] has found a widespread use in facilitating and dealing with different engineering challenges. On the basis of using fractional order dynamics[2], effective solutions have been proposed for some engineering problems[3] in different fields such as control system design[4−5], system identification[6−7], analysis and synthesis of electrical circuits[8−10], image and signal processing[11−12], robotics[13], electromagnetics[14], biomedical informatics[15], vibration reduction[16−17], wave propagation[18], and viscoelasticity[19].

In practice, sometimes there is a need to approximate fractional order systems. Consequently till now, different useful methods have been proposed for approximating fractional operators (for some sample methods, see [20−26]). But a main concern in using approximation methods is that the significant properties of the original fractional order systems may not be preserved after approximation[27−29]. Considering the importance of preserving the properties of fractional order systems in approximation process, some studies on this subject have been done in literature. For example, the problem of stability preservation has been investigated in [30−31] for methods presenting rational continuous-time filters for approximation of fractional operators. Also, the approximation methods constructed based on the direct discretization of fractional operators have been analyzed in the viewpoint of stability preservation in [32]. In this paper, the aim is to investigate the problem of response monotonicity preserving in approximation of fractional order systems by using rational approximations of fractional operators. Monotonicity of the step response is known as a feature for dynamical systems having desired transient responses[33−37]. Also, a necessary condition to have desired transient response in linear time invariant dynamical systems is monotonicity of the magnitude-frequency response[38]. Considering these points, dynamical systems with monotonic time/frequency responses have been taken into consideration in different applications[33−40]. In these applications, if we deal with a fractional order dynamical system with monotonic time/frequency response (for example in fractional order control system design or in fractional order filter synthesis with the aim of achieving a desired transient response)[41−43], approximating such a system may be unavoidable in practice. In this case, due to the significance of the property of response monotonicity, preserving such a property in approximation process is of great importance. In this paper, general conditions on rational approximations of fractional operators are derived that guarantee the preservation of monotonicity property of the step response or the magnitude-frequency response in approximation process.

The paper is organized as follows. In Section II, some preliminaries on approximation of fractional order systems are presented. Conditions for guaranteeing the preservation of the monotonicity property of the step response and the magnitude-frequency response in approximation of fractional order systems are respectively obtained in Sections III and IV. In these sections, numerical simulation results are also presented to confirm the usefulness of the obtained conditions. Finally, conclusions in Section V close the paper.

II. Rational Approximation of Fractional Order Systems

Consider a SISO LTI fractional order system described by the following pseudo-state space equations

$$\left\{\begin{array}{l} {}_0D_t^{\alpha}x(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{array}\right. \tag{1}$$

where $u(t) \in \mathbf{R}$, $y(t) \in \mathbf{R}$, and $x(t) \in \mathbf{R}^n$ are respectively the input, output, and pseudo-state vector of the system[44]. Also, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times 1}$, $C \in \mathbf{R}^{1\times n}$, $\alpha \in (0,1)$, and ${}_0D_t^{\alpha}$ denotes the Caputo derivative operator defined by

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Mohammad Saleh Tavazoei is with the Electrical Engineering Department, Sharif University of Technology, Tehran 11365-9363, Iran (e-mail: tavazoei@sharif.edu).

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$$ {}_0D_t^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x'(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau, \quad \alpha \in (0,1). \tag{2} $$

It is worth noting that (1) is the fractional order counterpart of the integer order system

$$ \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t). \end{cases} \tag{3} $$

The transfer function from input $u(t)$ to output $y(t)$ in system (1) is given as follows (See Fig. 1).

$$ G(s^{\alpha}) = C\left(s^{\alpha}I - A\right)^{-1}B. \tag{4} $$

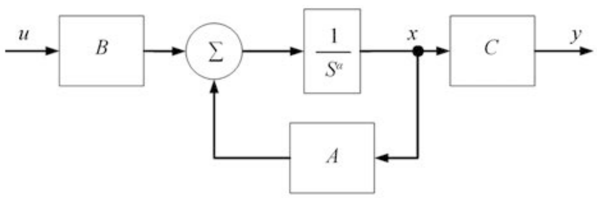

Fig. 1. Block diagram of pseudo-state space system (1).

A common way for approximating fractional order system (1) is to approximate and replace the operator $1/s^{\alpha}$ in block diagram of Fig. 1 by a rational transfer function (Fig. 2)[45]. Assume that the following approximation

$$ s^{\alpha} \approx P(s), \tag{5} $$

where

$$ P(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0}, \tag{6} $$

$a_i \geq 0$ for $i = 0, 1, \ldots, m-1$ and $b_i \geq 0$ for $i = 0, 1, ..., m$, is used for approximating fractional order system (1). In this case, the approximated system will be described by rational transfer function $G(P(s))$.

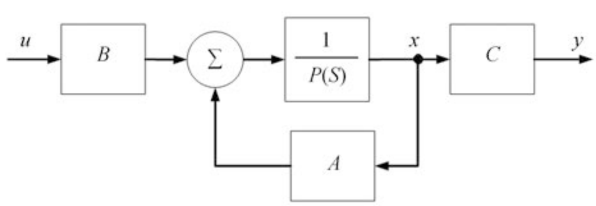

Fig. 2. Block diagram of the approximated integer order system for fractional order system (1) obtained by using approximation (5).

Preservation of the principal properties of the system is an important issue which should be taken into consideration in the approximation process. Considering this importance, on the basis of comparing the stability conditions of the original system (1) and its rational approximation, the stability preservation problem has been previously studied in [31–32]. In this paper, we focus on the monotonicity preservation problem for the time/frequency responses, and obtain conditions on approximation (5) for guaranteeing the preservation of the response monotonicity. Satisfying these conditions guarantees that the monotonicity of the system response is preserved in approximation process, i.e., the approximated integer order system similar as the original fractional order system has a monotonic time/frequency response. The main feature of the conditions obtained in the next sections is the independency of these conditions from the original system dynamics.

III. Step Response Monotonicity

A. Preliminaries

Consider a BIBO stable system described by transfer function $H(s)$. According to the Post-Widder inversion formula[46], the impulse response of this system $(h(t))$, as the inverse Laplace transform of $H(s)$, is given by

$$ h(t) = \lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k+1}{t}\right)^{k+1} H^{(k)}\left(\frac{k+1}{t}\right), \tag{7} $$

where $H^{(k)}(s)$ denotes the $k-th$ derivative of $H(s)$. An interesting consequence of the Post-Widder inversion formula, which has been taken into consideration in the literature[47], is about non-negativeness of the impulse response. According to this formula, if the BIBO stable transfer function $H(s)$ satisfies condition

$$ (-1)^r \, H^{(r)}(s)\Big|_{s=\sigma} \geq 0, \tag{8} $$

for r = 1, 2, ... and all positive real values of σ, then its impulse response is non-negative for $t > 0$. Conversely, by considering the Laplace transform definition, i.e.

$$ H(s) = \int_0^{\infty} \mathrm{e}^{-st} h(t)\,\mathrm{d}t, \tag{9} $$

it is deduced that

$$ (-1)^r H^{(r)}(\sigma) = \int_0^{\infty} \mathrm{e}^{-\sigma t} t^r h(t)\,\mathrm{d}t, \tag{10} $$

for $\sigma > 0$, where $H(s)$ is a BIBO stable transfer function with the impulse response $h(t)$. Therefore, if $h(t)$ is non-negative, then condition (8) is satisfied for r = 1, 2, ..., and all positive real values of σ[47].

B. Criteria for Step Response Monotonicity

This subsection deals with finding conditions on approximation filter (6) to guarantee monotonicity of the step response. Firstly, the monotonicity condition is derived in a general case (Theorem 1), and then this condition is simplified in some special cases (Corollaries 1-3). In addition, numerical examples are presented to validate the obtained results.

For obtaining the results in this subsection, it is assumed that the original system $(G(s^{\alpha}))$, the approximating filter $(P(s))$, and the approximated system $(G(P(s)))$ are BIBO stable (the conditions on approximating filter $(P(s))$ to preserve the stability of the system in the approximation process can be found in [31]). Now, as a first result consider the following theorem presenting a condition to guarantee the step response monotonicity in the approximation process.

Theorem 1. Assume that system (1) has a monotonic non-decreasing step response. The monotonicity of the step response is preserved by using rational approximation (5) if

$$ (-1)^k \, \tilde{P}^{(k)}(s)\Big|_{s=\sigma} \leq 0, \tag{11} $$

for all positive real values of σ and $k \in \mathbf{N}$, where $\tilde{P}(s) = P^{1/\alpha}(s)$.

Proof. Since the BIBO stable system (1) has a monotonic non-decreasing step response, according to discussions of Section III-A

$$(-1)^k \left. \hat{G}^{(k)}(s)\right|_{s=\sigma} \geq 0, \tag{12}$$

for all $k \in \mathbf{N}$ and $\sigma > 0$ where $\hat{G}(s) = G(s^\alpha) = C(s^\alpha I - A)^{-1}B$. By using the rational approximation (5), the approximated system is described by transfer function $\hat{G}(P^{1/\alpha}(s)) = \hat{G}(\tilde{P}(s))$. According to the Faàdi Bruno's formula (generalization of the chain rule for higher derivatives)[48], the r-th ($r \in \mathbf{N}$) derivative of this transfer function with respect to is given by

$$\frac{\mathrm{d}^r}{\mathrm{d}s^r}\hat{G}\left(\tilde{P}(s)\right) = \sum \frac{r!}{k_1!\ k_2!\ \cdots\ k_r!1!^{k_1}2!^{k_2}\cdots r!^{k_r}} \times \left.\hat{G}^{(k_1+k_2+\cdots+k_r)}(x)\right|_{x=\tilde{P}(s)} \prod_{i=1}^{r}\left(\tilde{P}^{(i)}(s)\right)^{k_i}, \tag{13}$$

where the sum appeared in the right-hand side of (13) is over all r-tuples of non-negative integers satisfying the Diophantine equation

$$k_1 + 2k_2 + \cdots + rk_r = r. \tag{14}$$

The Diophantine equation (14) yields in $\prod_{i=1}^{r}\left((-1)^i\tilde{P}^{(i)}(s)\right)^{k_i} = (-1)^r \prod_{i=1}^{r}\left((-1)^i\tilde{P}^{(i)}(s)\right)^{k_i}$. From this equality and (13),

$$(-1)^r\frac{\mathrm{d}^r}{\mathrm{d}s^r}\hat{G}\left(\tilde{P}(s)\right) = \sum \frac{r!}{k_1!\ k_2!\ldots k_r!1!^{k_1}2!^{k_2}\cdots r!^{k_r}} \times \left.\hat{G}^{(k_1+k_2+\cdots+k_r)}(x)\right|_{x=\tilde{P}(s)} \prod_{i=1}^{r}\left((-1)^i\tilde{P}^{(i)}(s)\right)^{k_i}. \tag{15}$$

If condition (11) is satisfied for all $\sigma > 0$ and $k \in \mathbf{N}$, then $\left(\left.(-1)^i\ \tilde{P}^{(i)}(s)\right|_{s=\sigma}\right)^{k_i}$ and $(-1)^{k_i}$ have the same signs for each $\sigma > 0$ and non-negative integer k_i. Considering this fact and (15), it is deduced that $(-1)^r \left.\frac{\mathrm{d}^r}{\mathrm{d}s^r}\hat{G}\left(\tilde{P}(s)\right)\right|_{s=\sigma}$ and $(-1)^{k_1+k_2+\cdots+k_r}\left.\hat{G}^{(k_1+k_2+\cdots+k_r)}(x)\right|_{x=\tilde{P}(\sigma)}$ have the same signs for all $\sigma > 0$. Hence, from (12) it is found that

$$(-1)^r \left.\frac{\mathrm{d}^r}{\mathrm{d}s^r}\hat{G}\left(\tilde{P}(s)\right)\right|_{s=\sigma} \geq 0, \tag{16}$$

for all $r \in \mathbf{N}$ and $\sigma > 0$. According to (16) and the Post-Widder inversion formula, it is concluded that the BIBO stable system $\hat{G}(\tilde{P}(s))$ (approximated system) has a monotonic non-decreasing step response. □

If $\alpha = 1/N$ where $N \in \mathbf{N}$, condition (11) is written as

$$(-1)^k \left.\left(P^N(s)\right)^{(k)}\right|_{s=\sigma} \leq 0. \tag{17}$$

According to discussions of Section III-A, condition (17) is satisfied for all $k \in \mathbf{N}$, if the BIBO stable transfer function $-P^N(s)$ has a non-negative impulse response for $t > 0$. Consequently, since the impulse response of this transfer function is equal to the negative of the impulse response of transfer function $P^N(s)$, the following corollary is deduced from Theorem 1.

Corollary 1. Assume that $\alpha = 1/N$ ($N \in \mathbf{N}$), and the step response of system (1) is monotonic non-decreasing. In this case, the monotonicity of the step response of system (1) is preserved by using rational approximation (5) if the rational transfer function $P^N(s)$ has a non-positive impulse response for $t > 0$.

Example 1. Consider the approximation $s^{0.5} \approx P(s)$ with (18), Shown at the bottom of the page, which is obtained by the low-frequency continued fraction method[49]. It can be verified that $P^2(s)$ has a non-positive impulse response for $t > 0$ (See Fig. 3). Hence, from Corollary 1 it is concluded that the step response monotonicity is preserved by using the above-mentioned approximation to approximate each fractional order system in the form (1) with α = 1/2 and a monotonic step response. For example, system (1) with α = 1/2, $A = \begin{bmatrix} -1.8 & -1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}}$, and $C = [0\ 1]$ is a BIBO stable system having a monotonic step response. As shown in Fig. 4, the monotonicity of the step response of this system is preserved by using the above-mentioned approximation.

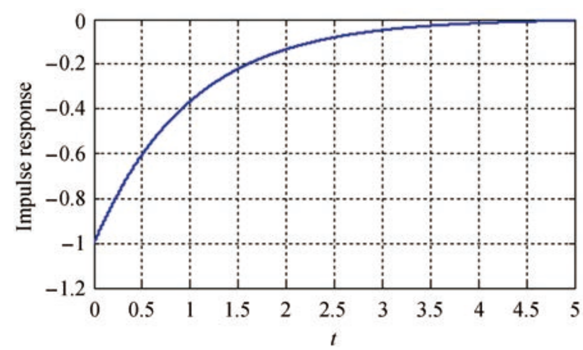

Fig. 3. Impulse response of $P^2(s)$ for $t > 0$ where $P(s)$ is defined by (18).

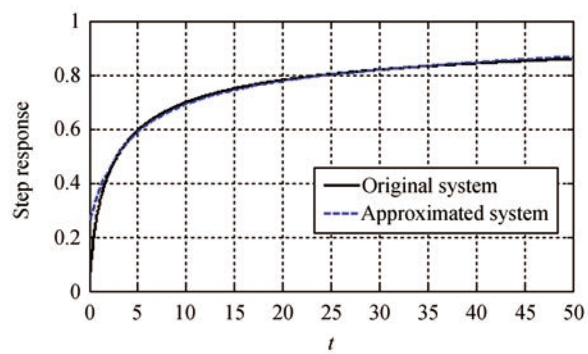

Fig. 4. Monotonic step responses of the original system and its approximation in Example 1.

$$P(s) = \frac{s(s+0.933)(s+0.75)(s+0.5)(s+0.25)(s+0.06699)}{(s+0.983)(s+0.8536)(s+0.6294)(s+0.3706)(s+0.1464)(s+0.01704)}, \tag{18}$$

According to [41, Theorem 3], we know that if (3) is a BIBO stable system with a monotonic step response, then (1) also has a monotonic step response. Hence in such a case, the monotonicity condition is reduced as that stated in the following corollary.

Corollary 2. Assume that the integer order system described by (3) is a BIBO stable system with a monotonic non-decreasing step response. In such a case, the step response of the approximation of system (1) obtained by using (5), similar as the step response of the original system (1), is monotonic if $P(s)$ has a non-positive impulse response for $t > 0$.

It is worth noting that the simple condition presented in Corollary 2 can be satisfied by a large class of rational approximations having interlaced real zeros and poles (For example, the rational approximations proposed in [21, 25, 51]). To show this fact, assume that the transfer function $P(s)$ described by

$$P(s) = k\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{m}(s+p_i)}, \tag{19}$$

where

$$k > 0 \ \& \ 0 \le z_1 < p_1 < z_2 < p_2 < \cdots < z_m < p_m \tag{20}$$

is used for approximation of fractional operator s^α ($\alpha \in (0,1)$). If condition (20) holds, $P(s)$ can be rewritten as

$$P(s) = k + \sum_{i=1}^{m}\frac{r_i}{s+p_i}, \tag{21}$$

where $r_i < 0$ for $i = 1, ..., m$ (See [50]). Since all r_i ($i = 1, ..., m$) are negative, transfer function (19) has a non-positive impulse response for $t > 0$. Therefore, the following result is deduced.

Corollary 3. Let assumptions of Corollary 2 hold. Then, the monotonicity of step response is preserved by using approximation (5) if $P(s)$ is in the form (19) and satisfies conditions in (20).

For instance, the approximation methods proposed in [21, 25, 51] satisfy conditions of Corollary 3. Consequently, using these methods results in preservation of the step response monotonicity in approximation of fractional order systems having monotonic step responses.

Example 2. In [41, Example 1], the monotonicity of the step response of a fractional order system is shown. The pseudo-state space representation of the system considered in [41, Example 1] is in the form (1) with the following matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -0.1 & -1.05 & -4.55 & -10.53 & -14.07 & -11.05 & -4.9 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}},$$
$$C = \begin{bmatrix} 2 & 1 & 2 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The step response of this system in the case $\alpha = 0.9$ is shown in Fig. 5. If the approximation

$$s^{0.9} \approx \frac{501.1872(s+0.001259)(s+0.1259)(s+12.59)}{(s+0.07943)(s+7.943)(s+794.3)}, \tag{22}$$

obtained based on the CRONE method[51], is used for approximating the above-described system, according to Corollary 3 the monotonicity of the step response is preserved. The step response of the approximated system shown in Fig. 5 confirms this point.

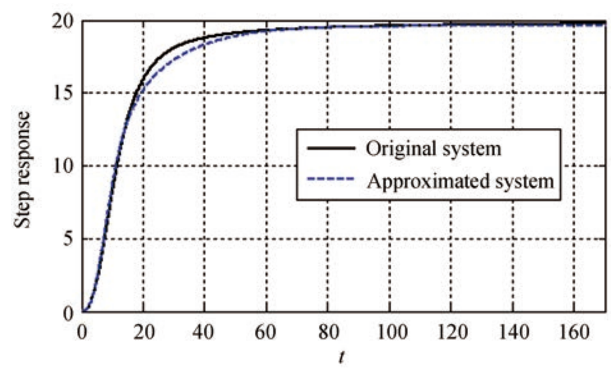

Fig. 5. Monotonic step responses of the original system and its approximation in Example 2.

IV. Magnitude-Frequency Response Monotonicity

The monotonicity of magnitude-frequency response of all-pole fractional order systems has been studied in [42]. In the mentioned study, algebraic conditions have been derived to guarantee the nonexistence of extrema in the magnitude-frequency response of an all-pole fractional order system. In continuation of the work done in [42], in this section it is assumed that (1) describes an all-pole transfer function in the form

$$G(s) = \frac{1}{\sum_{k=0}^{n} d_k s^{k\alpha}}, \quad d_n > 0 \ \& \ d_k \ge 0 \text{ for } k = 0, ..., n-1 \tag{23}$$

with the monotonic magnitude-frequency response $|G(\mathrm{j}\omega)|$ for $\omega \in (0,\infty)$. Considering this assumption, the following theorem presents conditions on the approximating filter $P(s)$ in (5) to preserve the monotonicity of the magnitude-frequency response of system (1) in the approximation process.

Theorem 2. The magnitude-frequency response monotonicity of any fractional order system in the form (1) with transfer function (23), which has a monotonic non-increasing magnitude-frequency response, is preserved in the frequency range (ω_i, ω_h) by approximating this system on the basis of approximation (5) if the following sets of conditions

$$\begin{cases} \min\limits_{c\in C^+} \frac{\Psi_c(\omega)}{\cos(\frac{c\alpha\pi}{2})} \ge 0, \\ \min\limits_{c\in C^+} \frac{\Psi_c(\omega)}{\cos(\frac{c\alpha\pi}{2})} \ge \max\limits_{c\in C^-} \frac{\Psi_c(\omega)}{\cos(\frac{c\alpha\pi}{2})}, \end{cases} \quad \forall \omega \in (\omega_l, \omega_h) \tag{24}$$

are satisfied where

$C^+ = \{c \in \{-n, ..., n\} |\ \cos(\frac{c\alpha\pi}{2}) \ge 0\}$,

$C^- = \left\{ c \in \{-n, ..., n\} \middle| \ \cos(\frac{c\alpha\pi}{2}) < 0 \right\}$, and

$$\Psi_c(\omega) = \frac{\mathrm{d}\,|P(\mathrm{j}\omega)|}{\mathrm{d}\omega} \cos\left(c\angle P(\mathrm{j}\omega)\right) - \frac{\mathrm{d}\angle P(\mathrm{j}\omega)}{\mathrm{d}\omega} \left|P(\mathrm{j}\omega)\right| \sin\left(c\angle P(\mathrm{j}\omega)\right). \quad (25)$$

Proof. It can be shown that if the magnitude-frequency response of transfer function (23) is monotonic non-increasing, then

$$f(\omega) \geq 0, \quad \forall \omega \in (0, \infty), \quad (26)$$

where $f(\omega)$ is defined as follows (For more details, see [42: Section 3]).

$$f(\omega) = \sum_{k=0}^{n} \sum_{l=0}^{n} k d_k d_l \cos\left(\frac{(k-l)\alpha\pi}{2}\right) \omega^{k+l-1}. \quad (27)$$

On the other hand, by using approximation (5) the magnitude-frequency response of the approximated system $G(P(s))$ is given by

$$\left|G\left(P(\mathrm{j}\omega)\right)\right| = \frac{1}{\sqrt{\rho(\omega)}}, \quad (28)$$

where

$$\rho(\omega) = \left(\sum_{k=0}^{n} d_k \left|P(\mathrm{j}\omega)\right|^k \cos\left(k\angle P(\mathrm{j}\omega)\right)\right)^2 + \left(\sum_{k=0}^{n} d_k \left|P(\mathrm{j}\omega)\right|^k \sin\left(k\angle P(\mathrm{j}\omega)\right)\right)^2. \quad (29)$$

The magnitude-frequency response of the approximated system is monotonic non-increasing in the frequency range (ω_i, ω_h) if and only if $\frac{\mathrm{d}\rho(\omega)}{\mathrm{d}\omega} \geq 0$, $\forall \omega \in (\omega_i, \omega_h)$. According to (29),

$$\begin{aligned}
&\frac{\mathrm{d}\rho(\omega)}{\mathrm{d}\omega} \\
&= 2\left(\sum_{k=0}^{n} d_k \left|P(\mathrm{j}\omega)\right|^k \cos\left(k\angle P(\mathrm{j}\omega)\right)\right) \\
&\quad \times \left(\begin{array}{c} \sum_{k=0}^{n} k d_k \left|P(\mathrm{j}\omega)\right|^{k-1} \frac{\mathrm{d}|P(\mathrm{j}\omega)|}{\mathrm{d}\omega} \cos\left(k\angle P(\mathrm{j}\omega)\right) \\ - k d_k \left|P(\mathrm{j}\omega)\right|^k \frac{\mathrm{d}\angle P(\mathrm{j}\omega)}{\mathrm{d}\omega} \sin\left(k\angle P(\mathrm{j}\omega)\right) \end{array}\right) \\
&\quad + 2\left(\sum_{k=0}^{n} d_k \left|P(\mathrm{j}\omega)\right|^k \sin\left(kP(\mathrm{j}\omega)\right)\right) \\
&\quad \times \left(\begin{array}{c} \sum_{k=0}^{n} k d_k \left|P(\mathrm{j}\omega)\right|^{k-1} \frac{\mathrm{d}|P(\mathrm{j}\omega)|}{\mathrm{d}\omega} \sin\left(k\angle P(\mathrm{j}\omega)\right) \\ + k d_k \left|P(\mathrm{j}\omega)\right|^k \frac{\mathrm{d}\angle P(\mathrm{j}\omega)}{\mathrm{d}\omega} \cos\left(k\angle P(\mathrm{j}\omega)\right) \end{array}\right).
\end{aligned} \quad (30)$$

By some calculations, (30) is simplified as

$$\frac{\mathrm{d}\rho(\omega)}{\mathrm{d}\omega} = \sum_{k=0}^{n} \sum_{l=0}^{n} k d_k d_l \left|P(\mathrm{j}\omega)\right|^{k+l-1} \Psi_{k-l}(\omega). \quad (31)$$

It is worth noting that condition (26) results in

$$f(|P(\mathrm{j}\omega)|) = \sum_{k=0}^{n} \sum_{l=0}^{n} k d_k d_l \cos\left(\frac{(k-l)\alpha\pi}{2}\right) \left|P(\mathrm{j}\omega)\right|^{k+l-1} \geq 0, \quad (32)$$

for $\omega \in (0, \infty)$. Define

$$\mu(\omega) = \min_{c \in C^+} \frac{\Psi_c(\omega)}{\cos(\frac{c\alpha\pi}{2})}, \quad \omega \in (\omega_l, \omega_h). \quad (33)$$

Definition (33) yields in

$$\Psi_c(\omega) \geq \mu(\omega) \cos(\frac{c\alpha\pi}{2}), \quad \forall \omega \in (\omega_l, \omega_h) \ \& \ \forall c \in \mathbf{C}^+. \quad (34)$$

Also if the second part of the conditions in (24) is met, then

$$\Psi_c(\omega) \geq \mu(\omega) \cos(\frac{c\alpha\pi}{2}), \quad \forall \omega \in (\omega_l, \omega_h) \ \& \ \forall c \in \mathbf{C}^-. \quad (35)$$

According to (31), (34), and (35), we have

$$\frac{\mathrm{d}\rho(\omega)}{\mathrm{d}\omega} \geq \sum_{k=0}^{n} \sum_{l=0}^{n} k d_k d_l \cos\left(\frac{(k-l)\alpha\pi}{2}\right) \mu(\omega) \left|P(\mathrm{j}\omega)\right|^{k+l-1}, \quad (36)$$

for all $\omega \in (\omega_l, \omega_h)$. If the first part of conditions in (24) (i.e., $\mu(\omega) \geq 0, \forall \omega \in (\omega_l, \omega_h)$ is satisfied, from (32) and (36) it is concluded that $\mathrm{d}\rho(\omega)/\mathrm{d}\omega \geq 0, \forall \omega \in (\omega_l, \omega_h)$. Hence, if the conditions in (24) hold, the approximated system has a monotonic non-increasing magnitude-frequency response in the frequency range (ω_i, ω_h). □

Example 3. The following approximation (37), shown at the bottom of the page, which has been obtained by using the low-frequency continued fraction method[49]. For approximation (37), functions $\min_{c \in \{-1,0,1\}} \frac{\Psi_c(\omega)}{\cos(0.3c\pi)}$ and $\min_{c \in \{-1,0,1\}} \frac{\Psi_c(\omega)}{\cos(0.3c\pi)} / \max_{c \in \{-2,2\}} \frac{\Psi_c(\omega)}{\cos(0.3c\pi)}$ have been respectively plotted versus ω in Figs. 6 and 7. Plotting these functions specify that the conditions in (24) are simultaneously satisfied in the frequency range $(0.074, \infty)$ for $n = 2$. Hence, from Theorem 2 the monotonicity of the magnitude-frequency response is preserved in the frequency range $(0.074, \infty)$ by using approximation (37) in approximating each all-pole fractional order system in the form (1) with $\alpha = 0.6$ and $n = 2$ which has a monotonic magnitude-frequency response. As a sample, consider system (1) with order $\alpha = 0.6$, and matrices $A = \begin{bmatrix} -1/40 & -1/10 \\ 1/16 & 0 \end{bmatrix}$, $B = [1/4 \quad 0]^{\mathrm{T}}$, and $C = [0 \quad 2/5]$. In this case, (1) is an all-pole system with a monotonic magnitude-frequency response (See Fig. 8). Although the magnitude-frequency response of the approximation of this system, obtained on the basis of (37), is not monotonic for all frequencies, the monotonicity of the magnitude-frequency response is preserved in the frequency range $(0.074, \infty)$ (See Fig. 8).

According to (31), a trivial condition which results in the monotonicity of the magnitude-frequency response is non-negative-ness of $\Psi_c(\omega)$. Hence, the following result is deduced.

$$s^{0.6} \approx \frac{s(s+0.8961)(s+0.6405)(s+0.3316)(s+0.08734)}{(s+0.9811)(s+0.8075)(s+0.5167)(s+0.2199)(s+0.03037)}, \quad (37)$$

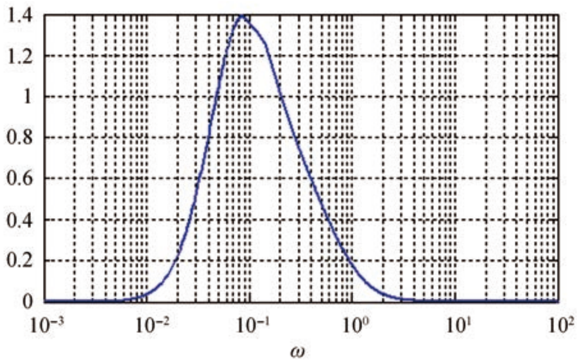

Fig. 6. $\min_{c\in\{-1,0,1\}} \frac{\Psi_c(\omega)}{\cos(0.3c\pi)}$ versus ω (Example 3).

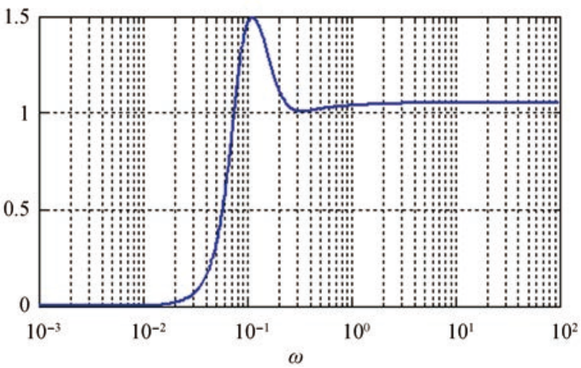

Fig. 7. $\min_{c\in\{-1,0,1\}} \frac{\Psi_c(\omega)}{\cos(0.3c\pi)} / \max_{c\in\{-2,2\}} \frac{\Psi_c(\omega)}{\cos(0.3c\pi)}$ versus ω (Example 3).

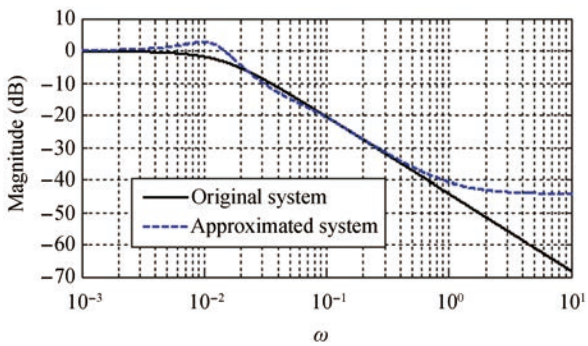

Fig. 8. Magnitude-frequency responses of the original system and its approximation in Example 3.

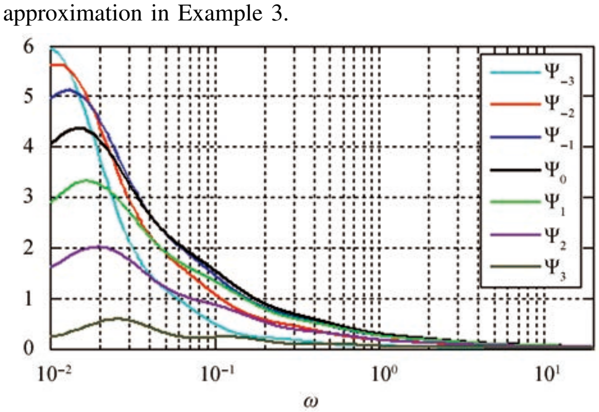

Fig. 9. $\Psi_c(\omega)$ versus ω for $c \in -3, \dots, 3$ (Example 4).

Corollary 4. If $\Psi_c(\omega) \geq 0$ for all $\omega \in (\omega_i, \omega_h)$ and $c \in -n, ..., n$, then the approximation of system (1) with transfer function (23), which is obtained on the basis of (5), is monotonic non-increasing in the frequency range (ω_i, ω_h).

Example 4. Approximation (38), shown at the bottom of the page, which is obtained by using the CRONE method [51, Sec. 4.1.1]. For this approximation, functions $\Psi_c(\omega)$ for $c \in -3, ..., 3$ are non-negative in the frequency range $(0, \infty)$ (See Fig. 9). Therefore according to Corollary 4, using approximation (38) in approximating system (1) with a transfer function in the form

$$G(s) = \frac{1}{d_3 s^{0.9} + d_2 s^{0.6} + d_1 s^{0.3} + d_0}, \tag{39}$$

where $d_k \geq 0$ for $k = 0, \dots, 3$, results in an approximated system with a monotonic magnitude-frequency response (According to [42, Corollary 1], transfer function (39) with condition $d_k \geq 0$ for $k = 0, \dots, 3$ has a monotonic magnitude-frequency response). For instance if approximation (38) is used for approximating the system

$$\begin{cases} {}_0D_t^{0.3}x(t) = \begin{bmatrix} -2 & -0.5 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(t), \end{cases} \tag{40}$$

as confirmed in Fig. 10 the approximated system similar as the original system has a monotonic magnitude-frequency response.

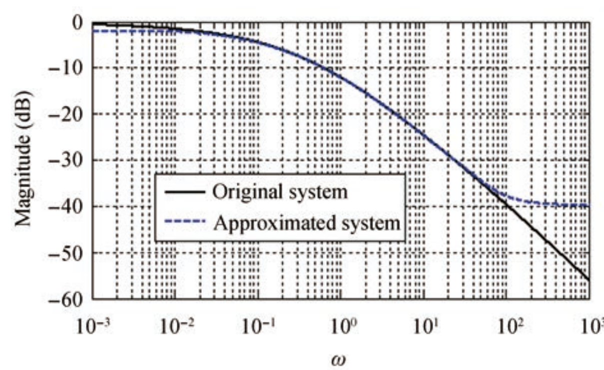

Fig. 10. Magnitude-frequency responses of the original system and its approximation in Example 4.

V. Conclusions

In this paper, the problem of preservation of response monotonicity in approximating fractional order systems by using rational approximations of fractional operators was investigated. In this investigation, conditions on the rational approximation of fractional operators were found to guarantee the monotonicity of step/magnitude-frequency response in approximation process (Theorems 1 and 2). These conditions were also simplified in some special cases (Corollaries 1-4).

$$s^{0.3} \approx 3.9811 \frac{(s + 0.01711)(s + 0.07943)(s + 0.3687)(s + 1.711)(s + 7.943)(s + 36.87)}{(s + 0.02712)(s + 0.1259)(s + 0.5843)(s + 2.712)(s + 12.59)(s + 58.43)}, \tag{38}$$

Moreover, numerical simulations results were presented to confirm the usefulness of the obtained conditions. The main significance of the conditions, which were obtained on approximating filters to guarantee the monotonicity of step/magnitude-frequency responses in approximation process, is the independency of these conditions from the original system dynamics. This means that if an approximating filter satisfies the obtained conditions, using this filter in approximation of each original system with a monotonic response results in an approximated system having a monotonic response. Generally speaking, this feature can considerably reduce the computational costs for investigating the problem of monotonicity preservation, where the aim is approximation of various fractional order systems having monotonic responses. In this case, the obtained conditions can be only checked for the approximating filter, and if these conditions are satisfied, monotonicity of the response is guaranteed for all the approximated systems resulted from using such an approximating filter. Proposing new monotonicity preserving methods for approximation of fractional order operators or determining the free parameters of the existing approximation methods to guarantee the preservation of the response monotonicity, on the basis of the conditions derived in this paper, can be considered as interesting topics for future research works.

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Mohammad Saleh Tavazoei received the B.Sc., M.Sc., and Ph.D. degrees in electrical engineering from the Sharif University of Technology, Tehran, Iran, in 2003, 2005, and 2008, respectively. He is currently an associate professor with the Department of Electrical Engineering, Sharif University of Technology. His current research interests include dynamical behavior analysis of fractional order systems and applications of these systems in control system design. Dr. Tavazoei is the recipient of the Young Investigator Award of IEEE Iran Section in 2012.