A Note on Robust Stability Analysis of Fractional Order Interval Systems by Minimum Argument Vertex and Edge Polynomials

Baris Baykant Alagoz

***Abstract*—By using power mapping ($s = v^m$), stability analysis of fractional order polynomials was simplified to the stability analysis of expanded degree integer order polynomials in the first Riemann sheet. However, more investigation is needed for revealing properties of power mapping and demonstration of conformity of Hurwitz stability under power mapping of fractional order characteristic polynomials. Contributions of this study have two folds: Firstly, this paper demonstrates conservation of root argument and magnitude relations under power mapping of characteristic polynomials and thus substantiates validity of Hurwitz stability under power mapping of fractional order characteristic polynomials. This also ensures implications of edge theorem for fractional order interval systems. Secondly, in control engineering point of view, numerical robust stability analysis approaches based on the consideration of minimum argument roots of edge and vertex polynomials are presented. For the computer-aided design of fractional order interval control systems, the minimum argument root principle is applied for a finite set of edge and vertex polynomials, which are sampled from parametric uncertainty box. Several illustrative examples are presented to discuss effectiveness of these approaches.**

***Index Terms*—Fractional order systems, robust stability, edge theorem, interval uncertainty.**

I. INTRODUCTION

ROBUST stability analysis is very essential for robust performance of practical control systems working in real applications. Imprecision in system modeling and temporal deviation of system parameters may cause instability of real control systems that are designed according to nominal system models. Implementation of practical and robust control systems requires the system design aspects, which ensure the stability of control systems within the possible ranges of system parameter fluctuations. Several theorems such as Kharitonov's theorem and edge theorem were developed to accomplish parametric robust stability analysis of integer-order system models introducing interval uncertainty of coefficients[1]. These theorems state sufficient conditions for robust stability and thus facilitate robust stability analyses of integer order linear time invariant (LTI) systems with parametric uncertainty. They limit stability checking to the certain number of polynomials sampled from a continuous family of interval characteristic polynomials. Nowadays, fractional order systems are on the focus of control community and confirmation of the validity of well-established robust stability analysis methods for fractional order system models is very beneficial. Robust stability analysis and robust stabilization problems of fractional order systems were addressed in many aspects during the last two decades[2−15]. It can be briefly explained as follows: Stability analysis of fractional order systems according to the pole placement in the complex plane was addressed by Matignon[15]. Minimum argument root principle was a milestone for robust stability analysis of fractional order interval systems. Based on minimum argument of eigenvalues of state space model, an interval boundary box method was presented for stability testing of the fractional order LTI systems with interval uncertainties[5]. Then, stabilization of fractional order LTI systems by using linear matrix inequality (LMI) method was shown in several works[6−9]. Robust stability check based on four Kharitonov's polynomials were also discussed for commensurate order LTI systems[10−11].

In many works, the power mapping of polynomials in complex planes, which is also known as conformal mapping, was employed to simplify stability analysis of fractional order systems by simply transforming them into integer order polynomials[3−4]. By applying $s = v^m$ mapping, stability analyses in the first Riemann sheet were shown for fractional order polynomials[13]. Later, a numerical method based on the exposed edge polynomial sampling was proposed for robust stability analysis of fractional order interval polynomials by using $s = v^m$ mapping[14]. However, it is obvious that there is a need for further works to demonstrate impacts of power mapping on root locus and stability related properties of systems. Thus, implication of edge theorem under power mapping of fractional order characteristic polynomials can be utilized and the robust stability analysis methods based on edge theorem can be developed to reduce computational complexity in robust stability analysis of fractional order interval systems.

In this paper, an investigation on the conformity of Hurwitz stability under power mapping ($s = v^m$) of characteristic polynomials of fractional order systems is presented. After showing that Hurwitz stability is conformal to power mapping, implications of edge theorem for fractional order characteristic polynomial are discussed. Then, robust stability analysis schemes considering combinations of edge and vertex polyno-

Manuscript received August 19, 2015; accepted February 28, 2016. Recommended by Associate Editor Dingyü Xue.

Citation: Baris Baykant Alagoz. A note on robust stability analysis of fractional order interval systems by minimum argument vertex and edge polynomials. *IEEE/CAA Journal of Automatica Sinica*, 2016, **3**(4): 411−421

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mials of the hyper-rectangle are presented and compared with the application of conventional edge theorem given in [14]. Computational complexity and effectiveness of presented methods are discussed by illustrative examples.

II. Basic Definitions and Preliminaries

Definition 1 (Hurwitz stability for integer order polynomials). Let us consider an integer order polynomial with real coefficient, which is expressed as $p(s)=\sum_{i=0}^{n}c_i s^i$. Parameters $c_i\in\mathbf{R}$ are real polynomial coefficients, and the parameter $n\in\mathbf{Z}^+$ represents the degree of the polynomials. A characteristic polynomial with real coefficients is said to be Hurwitz stable if and only if all of its roots lie in the left hand side of complex s plane[16−17]. Accordingly, Hurwitz stable polynomials are defined as $\{p(s)|p(s)=0:\forall s\in C\wedge \mathrm{Re}\{s\}<0\}$. If characteristic polynomial of a LTI system model is a Hurwitz stable polynomial, the LTI system model behaves asymptotically stable because time domain solutions consist of exponentially decaying terms. Consequently, root locus of characteristic polynomials has been widely used for the asymptotic stability analyses of LTI systems. The left hand side of complex plane, which is bounded by imaginary axis, is considered as the stability region for root locus analysis of integer order characteristic polynomials $p(s)$.

In general, the characteristic polynomial $p(s)$ is a multi-valued function of the complex variable s, whose domain is described by the principle sheet (the first sheet) of Riemann surfaces, defined in an argument range $-\pi<\arg(s)<\pi$[2]. As known, Hurwitz stability region (HSR) for characteristic polynomial roots is the left half plane of the first sheet, which can be defined according to root arguments as $\pi/2<\arg(s)<3\pi/2$. It is convenient to call the argument bounds with angles of $-\pi/2$ and $\pi/2$ as the Hurwitz stability boundary (HSB) for characteristic root locus[18]. The set of complex points with arguments $-\pi/2$ and $\pi/2$ also refers to the imaginary axis.

Definition 2 (Hurwitz stable fractional order polynomials). Let us consider a fractional order polynomial with real coefficients expressed in the form of $p_f(s)=\sum_{i=0}^{n}c_i s^{\alpha_i}$, where $\alpha_i\geq 0$ and $\alpha_i\in\mathbf{R}^+$ is the fractional orders of the polynomials. The case of $\alpha_0=0$ yields the constant term of polynomials. In order to facilitate root locus analysis of fractional order LTI systems, $s=v^m$ mapping has been used to transform a fractional order characteristic polynomial to the expanded degree integer order characteristic polynomials. It was shown by many works that one can carry out stability analysis of fractional order systems by examining root locus of the expanded degree integer-order characteristic polynomials, given as $p_f(s)|_{s=v^m}=p_m(v)=\sum_{i=0}^{n}c_i v^{m\alpha_i}$[2−3, 5−6, 13−14]. Here, each $m\alpha_i$ for $i=0,1,2,\dots,n$ is an integer number. Following the $s=v^m$ mapping, the first Riemann sheet is confined into a plane slice with the argument range $-\pi/m<\arg(v)<\pi/m$[2] and stability analyses were carried out in the first Riemann sheet[10−11, 13−14, 18]. In related works, by applying $s=v^m$ mapping, interval characteristic polynomials were recognized to be stable, in the case that all roots in the first Riemann sheet lie in complex plane slice with argument ranges of $(\pi/2m,\pi/m]$ and $[-\pi/m,-\pi/2m)$. Detailed works on the solution of fractional order characteristic polynomials were elaborated in [18] for analysis and design of control systems. Some useful properties of power mapping can be stated as follows:

Remark 1 (Magnitude and argument properties of power mapping). Let us consider the fraction order real polynomial $p_f(s)$, where complex input variable is defined as $s=M\mathrm{e}^{\mathrm{j}\theta}\in\mathbf{C}$. The $s=v^m$ transformation maps the function $p_f(s)$ to a real polynomial $p_m(v)$, where $v=\tilde{M}\mathrm{e}^{\mathrm{j}\phi}\in C$ such that the magnitude is $\tilde{M}=M^{(1/m)}$ and the argument is $\phi=\theta/m$.

Proof. This remark was previously mentioned in [3−4]. In order to better see mapping properties of $s=v^m$ transformation, one can write reverse transformation as $v=s^{1/m}=M^{(1/m)}\mathrm{e}^{\mathrm{j}\theta/m}$ for a complex point $s=M\mathrm{e}^{\mathrm{j}\theta}$, where parameters M and θ stand for the magnitude and argument of points in s domain. After rearranging $v=M^{(1/m)}\mathrm{e}^{\mathrm{j}\theta/m}$ in the form of $v=\tilde{M}\mathrm{e}^{\mathrm{j}\phi}$, it is obvious that $s=v^m$ transformation maps the complex point in s domain to a point in v domain, where the argument is $\phi=\theta/m$ and the magnitude is $\tilde{M}=M^{(1/m)}$. □

Remark 2 (Hurwitz stability region under $s=v^m$ mapping). An integer order characteristic polynomial is Hurwitz stable, if all characteristic roots lie in the left half side of complex plane. In case of $s=v^m$ mapping, the Hurwitz stability region is mapped into $(\pi/2m,\pi/m]$ for positive root arguments and $[-\pi/m,-\pi/2m)$ for negative root arguments.

Proof. Let us express Hurwitz stability region, given by $-\pi/2<\arg(s)<\pi/2$, as combination of $\pi/2<\theta<\pi$ for positive root arguments and $-\pi<\theta<-\pi/2$ for negative root arguments. Here, θ is the argument of a point in s domain and written as $\theta=\arg(s)$. By considering root argument transformation $\phi=\theta/m$ in Remark 1, Hurwitz stability region is mapped to $(\pi/2m,\pi/m]$ for positive root arguments and $[-\pi/m,-\pi/2m)$ for negative root arguments under $s=v^m$ mapping. Fig. 1 depicts the mapping of Hurwitz stability region under $s=v^m$ mapping for stability analysis. □

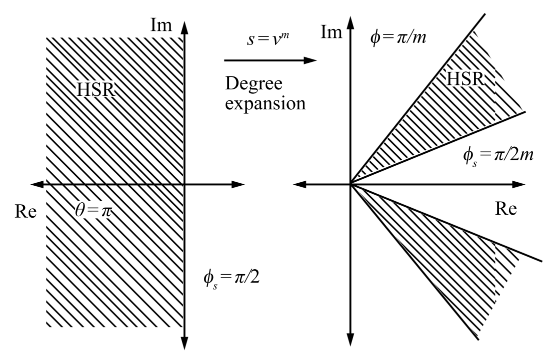

Fig. 1. Hurwitz stability region of fractional order characteristic polynomials under $s=v^m$ mapping[4].

Lemma 1 (Conservation of argument and magnitude relations). $s=v^m$ mapping of real polynomials is conservative in term of argument and magnitude relations. In other words, $s=v^m$ mapping conserves spatial relations between roots of s domain while mapping in v domain. For $m\in\mathbf{Z}^+$, the following root argument relations are valid,

$$\theta_i > \theta_j, \quad \theta_l < \theta_k, \quad \theta_u = \theta_q,$$
$$\Rightarrow \phi_i > \phi_j, \quad \phi_l < \phi_k, \quad \phi_u = \phi_q, \tag{1}$$

and the following root magnitude relations are also valid under $s = v^m$ mapping,

$$M_i > M_j, \quad M_l < M_k, \quad M_u = M_q,$$
$$\Rightarrow \tilde{M}_i > \tilde{M}_j, \quad \tilde{M}_l < \tilde{M}_k, \quad \tilde{M}_u = \tilde{M}_q. \tag{2}$$

Proof. For $m \in \mathbf{Z}^+$, one can write the following relations for arguments according to Remark 1,

$$\theta_i > \theta_j \rightarrow \frac{\theta_i}{m} > \frac{\theta_j}{m} \rightarrow \phi_i > \phi_j,$$
$$\theta_l > \theta_k \rightarrow \frac{\theta_l}{m} > \frac{\theta_k}{m} \rightarrow \phi_l > \phi_k,$$
$$\theta_u > \theta_p \rightarrow \frac{\theta_u}{m} > \frac{\theta_p}{m} \rightarrow \phi_u > \phi_p.$$

And the following relations for magnitudes,

$$M_i < M_j \rightarrow M_i^{(\frac{1}{m})} < M_j^{(\frac{1}{m})} \rightarrow \tilde{M}_i < \tilde{M}_j,$$
$$M_l > M_k \rightarrow M_l^{(\frac{1}{m})} > M_k^{(\frac{1}{m})} \rightarrow \tilde{M}_l > \tilde{M}_k,$$
$$M_u > M_q \rightarrow M_u^{(\frac{1}{m})} > M_q^{(\frac{1}{m})} \rightarrow \tilde{M}_u > \tilde{M}_q.$$

□

This lemma tells us that argument and magnitude relations are conserved under $s = v^m$ mapping.

III. Problem Statement

Fractional order LTI systems are represented by the fractional order differential equations in the form of [19],

$$a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \cdots + a_1 D^{\alpha_1} y(t) + a_0 y(t)$$
$$= b_m D^{\varphi_m} u(t) + b_{m-1} D^{\varphi_{m-1}} u(t) + \cdots$$
$$+ b_1 D^{\varphi_1} u(t) + b_0 u(t). \tag{3}$$

By using Laplace transform $L\{D^\alpha f(t)\} = s^\alpha F(s)$ for $f(0^+) = 0$[19], fractional order transfer functions are written to express system model in continuous frequency domain as follows.

$$T(s) = \frac{Y(s)}{U(s)} = \frac{\sum\limits_{i=0}^{m} b_i s^{\varphi_i}}{\sum\limits_{i=0}^{n} a_i s^{\alpha_i}}, \tag{4}$$

where denominator polynomial coefficients a_i and numerator polynomial coefficients b_i are real numbers. The fractional orders of the system are denoted by $\alpha_i \in \mathbf{R}$ ($i = 0, 1, 2, 3, \ldots, n$) and $\varphi_i \in \mathbf{R}$ ($i = 0, 1, 2, 3, \ldots, m$). For $\alpha_0 = 0$ and $\varphi_0 = 0$, the system models have constant terms a_0 and b_0. Here, the model orders satisfy $\alpha_n > \alpha_{n-1} > \alpha_{n-2} > \cdots > \alpha_2 > \alpha_1 > 0$ and $\varphi_n > \varphi_{n-1} > \varphi_{n-2} > \cdots > \varphi_2 > \varphi_1 > 0$.

In real systems, unpredictable parameter deviations and change of operating conditions lead to reduce consistency of system modeling. Therefore, a relevant modeling of real systems is not always possible to obtain by means of nominal LTI system models. The system modeling with parametric interval uncertainty is more convenient for the control system design problems compared to nominal system models. Because, systems can operate more effectively in real control application when controller performance is designed robust for possible ranges of system parameter variations. The characteristic polynomials of transfer functions with interval uncertainty are expressed as,

$$\Delta(s) = \sum_{i=0}^{n} [\underline{a}_i \, \bar{a}_i] s^{\alpha_i}, \tag{5}$$

where the parameters $[\underline{a}_i \, \bar{a}_i]$ represent uncertainty of the coefficient a_i, which refers to deviation between a lower ($\underline{a}_i$) and an upper ($\bar{a}_i$) bound. In practice, interval uncertainty bounds of the nominal coefficient a_i can be expressed by considering parameter deviation (Δa_i) as $[\underline{a}_i \, \bar{a}_i] = [a_i - \Delta a_i \;\; a_i + \Delta a_i]$. The checking of boundary conditions for robust stability is useful for control system design problems[18]. Example 3 is devoted to searching of the boundaries for allowable parameter deviation of robust stable control system according to edge theorem.

By applying $s = v^m$ to (5), one obtains expanded degree integer order characteristic polynomials in the form of,

$$\Delta^m(v) = \sum_{i=0}^{n} [\underline{a}_i \, \bar{a}_i] v^{\beta_i}, \tag{6}$$

where $\beta_i \in \mathbf{Z}^+ \cup \{0\}$ is expanded degree integer order, which is defined as $\beta_i = m\alpha_i$.

The uncertainty box of interval coefficients defines a hyper-rectangle (n-orthotope) denoted by A, which is written as Cartesian product of interval polynomial coefficients $\underline{a}_i \le a_i \le \bar{a}_i$, $i = 1, 2, 3, \ldots, n$,

$$A = \prod_{i=0}^{n} [\underline{a}_i \, \bar{a}_i]. \tag{7}$$

A point of the hyper-rectangle is represented by coefficient vector $a = [a_1 \, a_2 \, a_3 \cdots a_n]$. Each vector a from hyper-rectangle Astands for a fractional order polynomial from the interval polynomial family that is denoted by the set $\Omega \in \mathbf{R}^n$[14]. By considering real positive coefficient vectors, the hyper-rectangle A can be also expressed as $A = \{a : 0 < \underline{a}_i \le a_i \le \bar{a}_i, \; i = 1, 2, 3, \ldots, n\}$[14].

Let us assume that each characteristic polynomial from the set Ω has ξ numbers of complex roots, denoted by v_r in the complex v plane. The number of complex roots depends on the degree of $\Delta^m(v)$ and it can be found by $\xi = m\alpha_n$. The complex roots of the interval polynomial family Ω form a set of roots in the first Riemann sheet, which can be written as[14],

$$R(\Omega) = \{v_r : \Delta^m(a, v_r) = 0 \wedge |\arg(v_r)| < \frac{\pi}{m},$$
$$\forall a \in A, \; r = 1, 2, 3, \ldots, \xi\}. \tag{8}$$

Since expanded degree integer order characteristic polynomials are real coefficient integer order polynomials, complex roots lie symmetrically with respect to the real axis in complex v plane. The root region can be split into three subsets, $R(\Omega) = R(\Omega)^- \cup R(\Omega)^0 \cup R(\Omega)^+$, according to root arguments. Here, $R(\Omega)^+$, $R(\Omega)^-$ and $R(\Omega)^0$ represent subsets of $R(\Omega)$ formed with positive argument roots, negative argument roots and zero argument roots, respectively. The complex conjugate roots present symmetry at positive and negative argument sides[20], that is, one can state $|\arg(R(\Omega)^-)| \equiv |\arg(R(\Omega)^+)|$ for complex conjugate root arguments. Therefore, analysis given for positive arguments roots ($\phi > 0$) is also valid for negative argument roots ($\phi < 0$). This leads us to the conclusion; when $R(\Omega)^0$ is empty set, if only minimum positive argument characteristic roots lie in the stability region $(\pi/2m, \pi/m]$ in the first Riemann sheet under $s = v^m$ mapping, then the interval system can be recognized as Hurwitz stable[5, 15, 18].

$$\min\{\arg(R(\Omega))\} > \frac{\pi}{2m}. \tag{9}$$

Edge theorem is conformal under $s = v^m$ mapping due to Lemma 1. Because, conservation of argument and magnitude relations under power mapping ensures that root constellation in HSR of complex s plane is preserved in HSR of complex v plane and vice versa. Since argument relations of all edge and vertex roots are conserved under power mapping, edge theorem can be extended to complex v plane. On the other hand, due to the conservation of root argument relations according to Lemma 1, it is easy to see that if the minimum argument root in s domain lies in HSR region, it lies in HSR region of v domain. Fig. 2 depicts the mapping relations of the root constellation and vice versa. Lemma 1 also suggests us that vertex and edge of $R(\Omega)$ in v plane are also vertex and edge polynomials in s plane.

Theorem 1. (Conformity of minimum argument roots under power mapping): Under $s = v^m$ mapping ($m > 0$), theminimum argument root of a fractional order polynomial in s plane is mapped to the minimum argument root of its expanded integer order polynomials in v plane. Therefore, Hurwitz stability is conserved under power mapping.

Proof. Let us denote set of root arguments of all edges and vertex roots of $R(\Omega)$ as,

$$\psi = \arg(R(\Omega)) = \{\phi_1, \phi_2, \phi_3, \dots, \phi_k\}. \tag{10}$$

The minimum argument of root set $R(\Omega)$ is $\phi_{\min} = \min\{\arg(R(\Omega))\}$. If the condition $\phi_{\min} > \pi/2m$ is satisfied, the root set $R(\Omega)$ lies in Hurwitz stability region defined as $(\pi/2m, \pi/m]$ in Remark 2 due to the fact that $\forall \phi_i \in \psi$, $\phi_i \geq \phi_{\min} > \pi/2m$. One can rearrange it as $m\phi_i \geq m\phi_{\min} > \pi/2$, which refers to $\theta_i \geq \theta_{\min} > \pi/2$ where $\theta_{\min} = m\phi_{\min}$ and $\theta_i = m\phi_i$ according to Remark 1. Therefore, if the expanded degree integer order characteristic polynomials are Hurwitz stable, fractional order characteristic polynomial is also Hurwitz stable. It is shown for unstable cases in the same manner. One can state that the stability properties related with root locus are conformal under power mapping. □

It is noteworthy that the root region in v plane is indeed the scaled and rotated image of root region in s plane according to magnitudeand argument properties ($\tilde{M} = M^{(1/m)}$, $\phi = \theta/m$) given in Remark 1. A graph defined by edge and vertices roots on root region $R(\Omega)$ is preserved under power mapping.

IV. Implications of Edge Theorem with Minimum Argument Root Principle for Fractional Order Interval Polynomials

Edge theorem provides consistent solutions for the robust stability analyses of integer-order LTI interval systems[1]. For

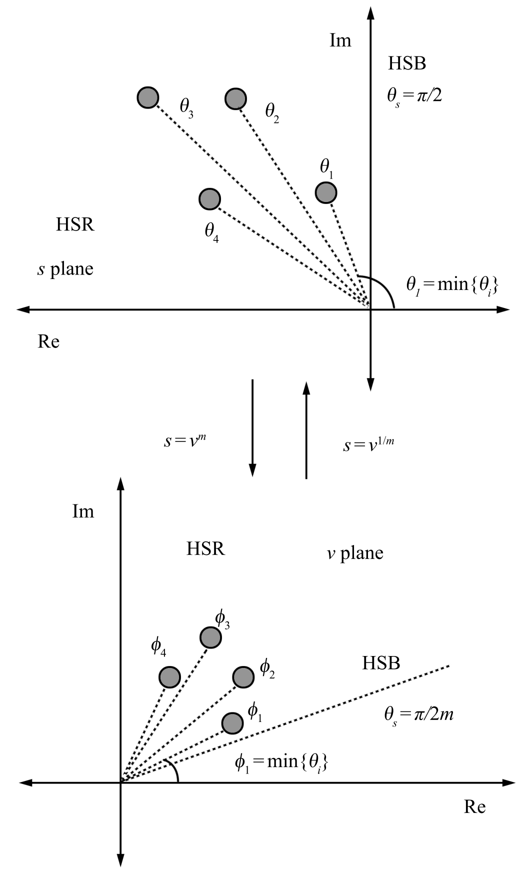

Fig. 2. Argument and magnitude relations are conformal under power mappings (Root placements in s and v planes conserve argument relations).

fractional order interval systems, an application of edge theorem for numerical robust stability analysis was demonstrated by Senol et al.[14]. The boundary of root region in the first Riemann sheet was represented by roots of exposed edge polynomials of interval coefficient hyper-rectangle. In this section, with consideration of minimum argument root principle for vertex and its connected edge polynomials, author aims to reduce computational complexity of the robust stability analysis of fractional order system models. Fig. 3 (a) depicts the exposed edge and vertex polynomials of hyper-rectangle A, which was drawn for the case of three interval coefficients. Fig. 3 (b) illustrates the corresponding roots of exposed edge and vertex polynomials of hyper-rectangle A in the first Riemann sheet. Fig. 3 (c) indicates the minimum argument vertex root and roots of its connected edges. This study suggests that analysis of roots in Fig. 3 (c) can significantly reduce computational complexity of robust stability analysis of fractional order interval polynomials.

Edge theorem also implies that boundary of root region is formed by roots of vertex and exposed edge polynomials of uncertainty box, which is represented by hyper-rectangle A of interval coefficients. It is obvious that the change of root location with respect to the change of polynomial coefficients is continuous due to the fact that polynomials and their coefficient intervals are continuous. According to this reason, the most outer roots of root region $R(\Omega)$ can come from edge and vertex polynomials of hyper-rectangle A. A given root region seen in Fig. 4 validates this effect. Boundaries of root region were formed by roots of vertex and exposed edges polynomials of A.

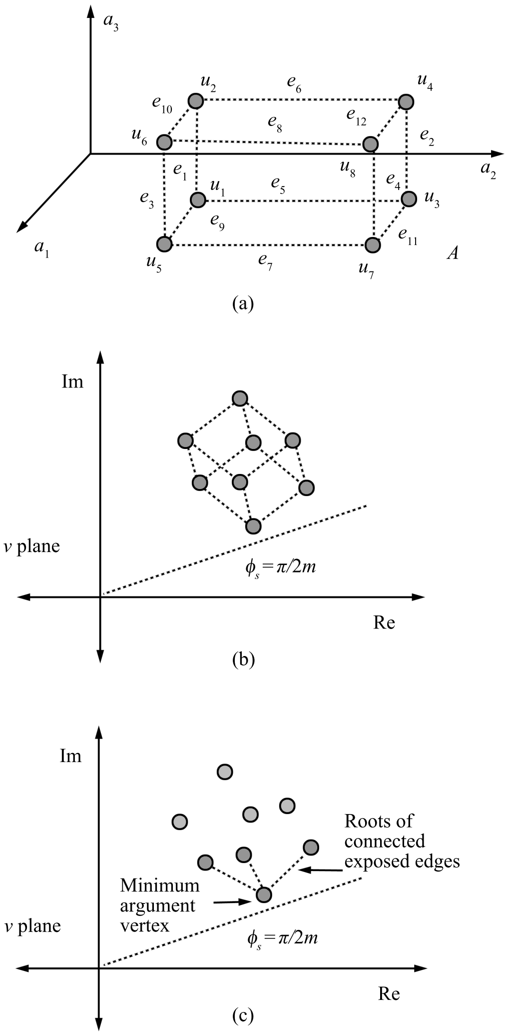

Fig. 3. Illustrations for three-dimensional hyper-rectangle and root placements ((a) An illustration of vertex $(u_1, u_2, u_3, \ldots, u_8)$ and exposed edges $(e_1, e_2, e_3, \ldots, e_8)$ of hyper-rectangle A build for three interval coefficients; (b) Root locus of exposed edge and vertex polynomials; (c) Root locus of minimum argument vertex polynomial and roots of connected exposed edge polynomials.)

As known, n number of uncertain parameters builds 2^n vertices on the hyper-rectangle. Coefficient vectors of vertex polynomials of A were expressed as Cartesian products of upper and lower bounds of interval coefficients[14],

$$\begin{aligned} u_k = \ & \{\underline{a}_0, \bar{a}_0\} \times \{\underline{a}_1, \bar{a}_1\} \times \{\underline{a}_2, \bar{a}_2\} \times \cdots \\ & \times \left\{\underline{a}_{n-1}, \bar{a}_{n-1}\right\} \times \{\underline{a}_n, \bar{a}_n\}, \end{aligned} \tag{11}$$

where "$\times$" represents Cartesian product operator. Let us express vertex polynomials of expanded degree integer order interval polynomials as,

$$\Delta_{u_k} = \Delta^m(u_k, v), \quad k = 1, 2, 3, \ldots, 2^n. \tag{12}$$

Exposed edges are line segments connecting vertices through the surfaces of A as illustrated in Fig. 3 (a). The edge polynomials can be obtained by sampling coefficient vectors from exposed edge of A[14],

$$\begin{aligned} e_k = & \{\underline{a}_0, \bar{a}_0\} \times \{\underline{a}_1, \bar{a}_1\} \times \{\underline{a}_2, \bar{a}_2\} \times \cdots \\ & \times s(a_k, \lambda) \times \cdots \times \left\{\underline{a}_{n-1}, \bar{a}_{n-1}\right\} \times \{\underline{a}_n, \bar{a}_n\} \end{aligned} \tag{13}$$

where $s(a_k, \lambda)$ is edge sampling function and defined linearly as $s(a_k, \lambda) = \lambda \underline{a}_k + (1-\lambda)\bar{a}_k$, $\lambda \in [0, 1]$. Edge polynomials of expanded degree integer order interval characteristic polynomial are expressed as,

$$\Delta_{e_k} = \Delta^m(e_k, v). \tag{14}$$

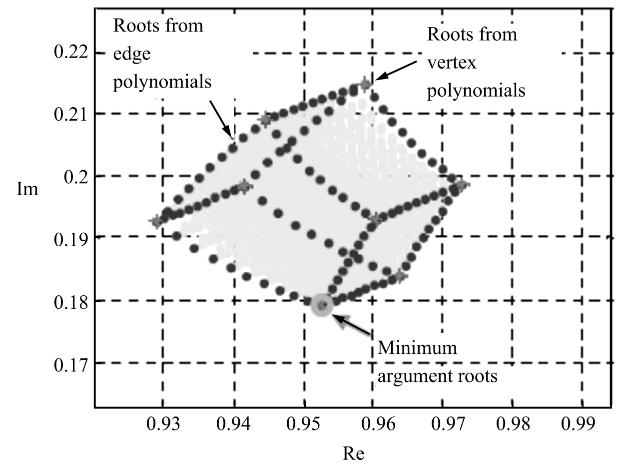

Fig. 4. Root region of expanded degree integer order interval polynomial $\Delta^{10}(v) = [2.1\ 2.6]v^{21} + [1.2\ 1.7]v^8 + [0.7\ 1.3]$ in the first Riemann sheet. Roots from edge polynomials and roots from vertex polynomials of hyper-rectangle A are indicated by blue dots and red asterisks, respectively.

It is unnecessary to check all edge polynomials for robust stability checking. Because, if one can show that minimum argument root lies in stability region, the interval system is robust stable. Computational complexity of robust stability analysis based on root locus strongly depends on the number of tested polynomials and the number of polynomials to be solved increases depending on the number of exposed edges of hyper-rectangle, which is expressed as $n2^{n-1}$, where n is the number of interval coefficients.

Our approach to reduce complexity of numerical analysis is to consider only connected exposed edge polynomials of the minimum argument vertex polynomial. The number of the connected edges of a vertex is n. Complexity reduction depending on edge number can be expressed depending on total edge counts as $G(n) = n/(n2^{n-1}) = 2^{-n+1}$. This indicates an exponential decay of complexity reduction depending on considered edge counts.

Set of vertex roots in the root region $R(\Omega)$ can be expressed as

$$\begin{aligned} R_u(\Omega) = \{v : \Delta_{u_k}(a, v) = 0 \wedge |\arg(v)| < \frac{\pi}{m}, \\ \forall a \in A,\ \ k = 1, 2, 3, \ldots, 2^n\}. \end{aligned} \tag{15}$$

Set of exposed edge roots in the root region $R(\Omega)$ can be expressed as

$$R_e(\Omega)=\{v:\Delta_{e_k}(a,v)=0\wedge|\arg(v)|<\frac{\pi}{m},\\ \forall a\in A,\ \ k=1,2,3,\ldots,n2^{n-1}\}.\quad(16)$$

Edge theorem suggests that stability checking of all exposed edge polynomials is sufficient to show robust stability of integer order interval characteristic polynomials. Considering this theorem and using power mapping, stability analyses according to test of polynomials taken from all exposed edge and vertex polynomials ($R_u(\Omega)\cup R_e(\Omega)$) were discussed in [14]. In the case of sampling edges with n_p polynomials ($n_p > 2$), the method in [14] requires test of $n_pn2^{n-1}+2^n$ polynomials. Here, n_pn2^{n-1} polynomials are for sampling of edges and 2^n polynomials are for vertex polynomials. In order to simplify robust stability analyses, the following two approaches are proposed in this study:

1) Test of minimum argument vertex with connected edge polynomials (MVCE): For positive interval coefficient polynomials, it can be possible to reduce number of test polynomials by only evaluating stability of the minimum argument vertex polynomial and its connected edge polynomials. It requires the calculation of roots from $R_u(\Omega)\cup R_{ce}(\Omega)$, where $R_{ce}(\Omega)\in R_e(\Omega)$ is a subset of edge roots. It needs only testing of the exposed edge polynomials connected to the minimum argument vertex that is defined as $\min\{\arg(R_u(\Omega))\}$. In the case of an edge polynomial sampling with n_p polynomials, MVCE approach requires the test of n_pn+2^n polynomials. This approach is valid under the assumption that the branch of edge graph, composed of the minimum argument vertex and its connected edges, includes the minimum argument root of A. Since the boundary of root region is formed by only roots of vertex and exposed edge polynomials of A, minimum argument root is the most likely to be on the minimum argument vertex or its connected edge polynomials.

2) Test of minimum argument vertex (MV): It is possible to reduce further the number of test polynomials by considering only vertex polynomials of hyper-rectangle A. This approach requires calculation of $\min\{\arg(R_u(\Omega))\}$, so it performs the test of 2^n polynomials. This test relies on the assumption that minimum argument roots probably come from vertex polynomials of hyper-rectangle because the interval polynomial coefficients are continuous and lead to continuity of root locus. The most distant polynomials of hyper-rectangle A are vertex polynomials. The roots of vertex polynomials of A form the vertices of root region.

Table I shows the number of the test polynomials required for robust stability analysis for edge theorem based approaches. Fig. 5 shows increase of test polynomials with respect to number of interval coefficients (n) for 20 polynomials edge sampling ($n_p = 20$). It can be seen that the test of minimum argument vertex polynomials (MV) is very advantageous in term of computational complexity.

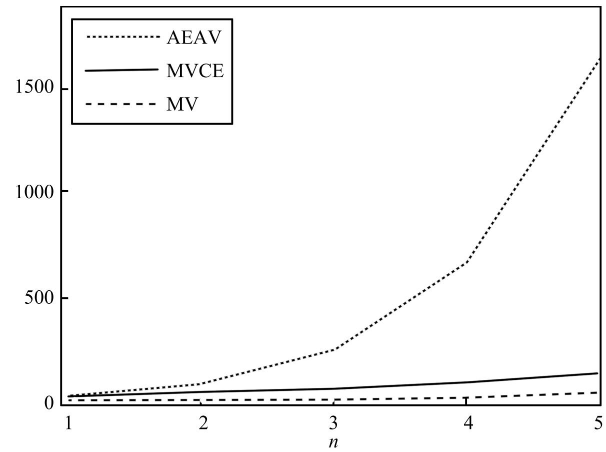

Fig. 5. Number of the tested polynomials required for test of all edge and vertex polynomials (AEAV), for the test of minimum argument vertex with connected edge polynomials (MVCE) and for the test of minimum argument vertex polynomials (MV).

V. ILLUSTRATIVE EXAMPLES

Initial conditions of systems were assumed to be zero for all parameters in numerical analyses.

Example 1. By considering the fractional order LTI nominal system described by fractional order differential equations[3],

$$0.8D^{2.2}y(t)+0.5D^{0.9}y(t)+y(t)=u(t).\qquad(17)$$

Let us check robust stability of this system for interval uncertainty of coefficients given as $0.8\pm0.4=[0.4\ 1.2]$, $0.5\pm0.2=[0.3\ 0.7]$ and $1\pm0.3=[0.7\ 1.3]$.

To simplify analysis of interval system, one can express it in the form of transfer function with zero initial conditions as

$$G(s)=\frac{Y(s)}{U(s)}=\frac{1}{[0.4\ 1.2]s^{2.2}+[0.3\ 0.7]s^{0.9}+[0.7\ 1.3]},\qquad(18)$$

TABLE I
ROBUST STABILITY ANALYSIS APPROACHES BASED ON EDGE THEOREM FOR REAL POSITIVE COEFFICIENT INTERVAL CHARACTERISTIC POLYNOMIALS

Approaches based on edge theorem	Number of test polynomials	Basic assumptions
Test of all edge and vertex polynomials (AEAV)[14]	$n_pn2^{n-1}+2^n$	The test of all exposed edge polynomials from A is sufficient to show robust stability
Test of minimum argument vertex with connected edge polynomials (MVCE)	n_pn+2^n	Set of minimum argument vertex with connected edge polynomials generally includes minimum argument root of A
Test of minimum argument vertex polynomials (MV)	2^n	Minimum argument root is probably the root of vertex polynomials of A

and the characteristic polynomial of the system is found as

$$\Delta(s) = [0.4\ 1.2]s^{2.2} + [0.3\ 0.7]s^{0.9} + [0.7\ 1.3]. \tag{19}$$

By applying $s = v^m$ mapping, the expanded degree integer order characteristic polynomial is written as,

$$\Delta^{10}(v) = [0.4\ 1.2]v^{22} + [0.3\ 0.7]v^{9} + [0.7\ 1.3]. \tag{20}$$

Vertex polynomials of expanded degree integer order interval characteristic polynomial were obtained as

$$\begin{aligned}\{\Delta_{u_1} &= \Delta^{10}([0.4\ 0.3\ 0.7], v),\ \Delta_{u_2} = \Delta^{10}([0.4\ 0.3\ 1.3], v),\\ \Delta_{u_3} &= \Delta^{10}([0.4\ 0.7\ 0.7], v),\ \Delta_{u_4} = \Delta^{10}([0.4\ 0.7\ 1.3], v),\\ \Delta_{u_5} &= \Delta^{10}([1.2\ 0.3\ 0.7], v),\ \Delta_{u_6} = \Delta^{10}([1.2\ 0.3\ 1.3], v),\\ \Delta_{u_7} &= \Delta^{10}([1.2\ 0.7\ 0.7], v),\ \Delta_{u_8} = \Delta^{10}([1.2\ 0.7\ 1.3], v)\}\end{aligned}$$

We performed edge sampling with 19 polynomials ($n_p = 19$) in numerical analyses, so edge sampling function can be written as $s(a_k, \lambda) = \lambda \underline{a}_k + (1-\lambda)\bar{a}_k$, where $\lambda \in \{0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45, 0.50, 0.55, 0.60, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95\}$ and $k = \{1, 2, 3\}$. Set of edge polynomials were obtained as,

$$\begin{aligned}\{\Delta_{e_1} &= \Delta^{10}([(\lambda 0.4 + (1-\lambda)1.2)\ 0.3\ 0.7], v),\\ \Delta_{e_2} &= \Delta^{10}([(\lambda 0.4 + (1-\lambda)1.2)\ 0.3\ 1.3], v),\\ \Delta_{e_3} &= \Delta^{10}([(\lambda 0.4 + (1-\lambda)1.2)\ 0.7\ 0.7], v),\\ \Delta_{e_4} &= \Delta^{10}([(\lambda 0.4 + (1-\lambda)1.2)\ 0.7\ 1.3], v),\\ \Delta_{e_5} &= \Delta^{10}([\ 0.4\ (\lambda 0.3 + (1-\lambda)0.7)\ 0.7], v),\\ \Delta_{e_6} &= \Delta^{10}([\ 0.4\ (\lambda 0.3 + (1-\lambda)0.7)\ 1.3], v),\\ \Delta_{e_7} &= \Delta^{10}([\ 1.2\ (\lambda 0.3 + (1-\lambda)0.7)\ 0.7], v),\\ \Delta_{e_8} &= \Delta^{10}([\ 1.2\ (\lambda 0.3 + (1-\lambda)0.7)\ 1.3], v),\\ \Delta_{e_9} &= \Delta^{10}([\ 0.4\ \ 0.3\ (\lambda 0.7 + (1-\lambda)1.3)], v),\\ \Delta_{e_{10}} &= \Delta^{10}([\ 0.4\ \ 0.7\ (\lambda 0.7 + (1-\lambda)1.3)], v),\\ \Delta_{e_{11}} &= \Delta^{10}([\ 1.2\ \ 0.3\ (\lambda 0.7 + (1-\lambda)1.3)], v),\\ \Delta_{e_{12}} &= \Delta^{10}([\ 1.2\ \ 0.7\ (\lambda 0.7 + (1-\lambda)1.3)], v)\}.\end{aligned}$$

Figs. 6 (a) and 6 (b) show roots of vertex and edge polynomials of A in the first Riemann sheet. Roots of vertex polynomials are indicated by blue asterisks. Roots of minimum argument vertex and connected edge polynomial are indicated by red dots in complex v plane. Minimum argument root is the root of vertex polynomial $\Delta_{u_6} = \Delta^{10}([1.2\ 0.3\ 1.3], v)$ and the value of minimum argument is $\phi_{\min} = \min\{\arg(R(\Omega))\} = 0.0487$ radian. Since it is lower than the stability boundary $\phi_s = \frac{\pi}{20}$, the interval system is not robust stable.

Fig. 7 shows step response of 8 vertex polynomials. The step response obtained for $u_6 = [1.2\ 0.3\ 1.3]$ confirms the unstable response of the interval system.

Example 2. By considering the closed loop control of electrical heater, which was modeled by fractional order plant function, $G(s) = \frac{Y(s)}{U(s)} = \frac{1}{39.96s^{1.25}+0.598}$, and the integer

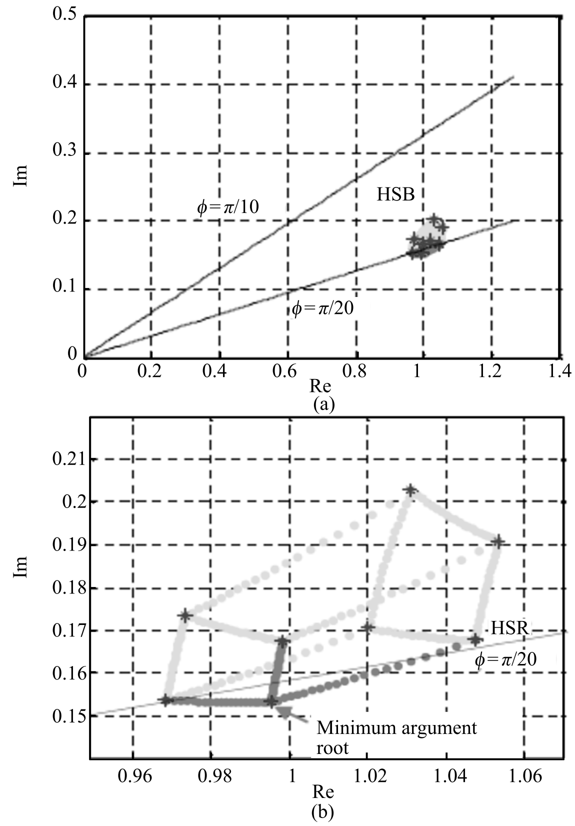

Fig. 6. Root placement for Example 1 ((a) Roots of vertex and edge polynomials; (b) Close view of minimum argument vertex and connected edge polynomials.)

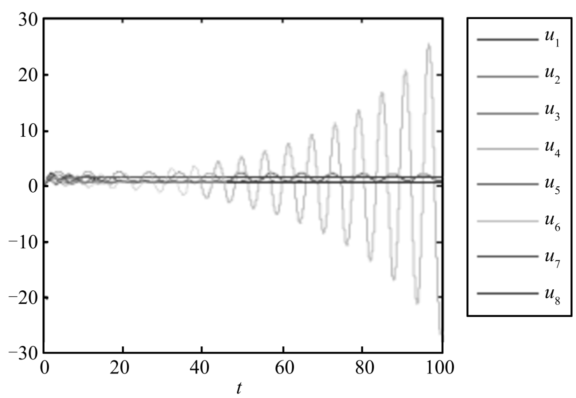

Fig. 7. Step responses of vertex polynomials.

order PD controller, $C(s) = 64.47 + 12.46s$[3], let us check robust stability of closed loop control system for interval uncertainty given as the following:

For electrical heater model, parameter deviations are 39.96 $\pm$ 5.3 = [34.66 45.26] and $0.58 \pm 0.12 = [0.46\ 0.7]$, and for the controller function, parameter deviations are 64.47 ± 11.5 = [52.97 75.97] and $12.46 \pm 3.36 = [9.10\ 15.82]$.

The resulting closed loop transfer function of interval system becomes,

$$T(s) = \frac{Y(s)}{U(s)} = \frac{[9.10\ \ 15.82]s + [52.97\ \ 75.97]}{[34.66\ \ 45.26]s^{1.25} + [9.10\ \ 15.82]s + [53.43\ \ 76.67]}, \tag{21}$$

and then characteristic polynomial of the system can be expressed as,

$$\Delta(s) = [34.66\ \ 45.26]s^{1.25} + [9.10\ \ 15.82]s + [53.43\ \ 76.67]. \tag{22}$$

By applying $s = v^{100}$ mapping, the expanded degree integer order characteristic polynomial is written as,

$$\Delta^{100}(v) = [34.66\ \ 45.26]v^{125} + [9.10\ \ 15.82]v^{100} + [53.43\ \ 76.67]. \tag{23}$$

Vertex polynomials of expanded degree integer order interval characteristic polynomial were obtained as

$$\begin{aligned}\{\Delta_{u_1} &= \Delta^{100}([34.66\ 9.10\ 53.43], v),\\ \Delta_{u_2} &= \Delta^{100}([34.66\ 9.10\ 76.67], v),\\ \Delta_{u_3} &= \Delta^{100}([34.66\ 15.83\ 53.43], v),\\ \Delta_{u_4} &= \Delta^{100}([34.66\ 15.83\ 76.67], v),\\ \Delta_{u_5} &= \Delta^{100}([45.26\ 9.10\ 53.43], v),\\ \Delta_{u_6} &= \Delta^{100}([45.26\ 9.10\ 76.67], v),\\ \Delta_{u_7} &= \Delta^{100}([45.26\ 15.82\ 53.43], v),\\ \Delta_{u_8} &= \Delta^{100}([45.26\ 15.82\ 76.67], v)\}.\end{aligned}$$

We used 19 polynomials edge sampling ($n_p = 19$) in numerical analyses. Set of edge polynomials can be written as,

$$\begin{aligned}\{\Delta_{e_1} &= \Delta^{100}([(\lambda 34.66 + (1-\lambda)45.26)\ 9.10\ 53.43], v),\\ \Delta_{e_2} &= \Delta^{100}([(\lambda 34.66 + (1-\lambda)45.26)\ 9.10\ 76.67], v),\\ \Delta_{e_3} &= \Delta^{100}([(\lambda 34.66 + (1-\lambda)45.26)\ 15.82\ 53.43], v),\\ \Delta_{e_4} &= \Delta^{100}([(\lambda 34.66 + (1-\lambda)45.26)\ 15.82\ 76.67], v),\\ \Delta_{e_5} &= \Delta^{100}([\ 34.66\ (\lambda 9.10 + (1-\lambda)\ 15.82)\ 53.43], v),\\ \Delta_{e_6} &= \Delta^{100}([\ 34.66\ (\lambda 9.10 + (1-\lambda)\ 15.82)\ 76.67], v),\\ \Delta_{e_7} &= \Delta^{100}([\ 45.26\ (\lambda 9.10 + (1-\lambda)\ 15.82)\ 53.43], v),\\ \Delta_{e_8} &= \Delta^{100}([\ 45.26\ (\lambda 9.10 + (1-\lambda)\ 15.82)\ 76.67], v),\\ \Delta_{e_9} &= \Delta^{100}([\ 34.66\ 9.10\ (\lambda 53.43 + (1-\lambda)\ 76.67)\ ], v),\\ \Delta_{e_{10}} &= \Delta^{100}([\ 34.66\ 15.82\ (\lambda 53.43 + (1-\lambda)\ 76.67)\ ], v),\\ \Delta_{e_{11}} &= \Delta^{100}([\ 45.26\ 9.10\ (\lambda 53.43 + (1-\lambda)\ 76.67)\ ], v),\\ \Delta_{e_{12}} &= \Delta^{100}([\ 45.26\ 15.82\ (\lambda 53.43 + (1-\lambda)\ 76.67)\ ], v)\}.\end{aligned}$$

Figs. 8 (a) and 8 (b) show roots of vertex and edge polynomials in the first Riemann sheet. Minimum argument root is the root of vertex polynomial $\Delta_{u_6} = \Delta^{100}([45.26\ \ 9.10\ \ 76.67],\ v)$ and the value of minimum argument is $\phi_{\min} = \min\{\arg(R(\Omega))\} = 0.0259$ radian. Since minimum argument root lies in HSR defined with the root argument interval $(\pi/200,\ \pi/100]$, the interval system is robust stable.

Fig. 9 shows step responses of 8 vertex polynomials and confirms robust stability of the closed loop electrical heater control system for the given parameter deviation ranges.

Example 3. By considering the closed loop electrical heater control system given in previous example as the plant function $G(s) = \frac{Y(s)}{U(s)} = \frac{1}{39.96s^{1.25}+0.598}$ and PD controller of system, $C(s) = 64.47 + 12.46s$. By using edge theorem approaches, let us find out interval uncertainly ranges of γ that make the closed loop control system robust stable.

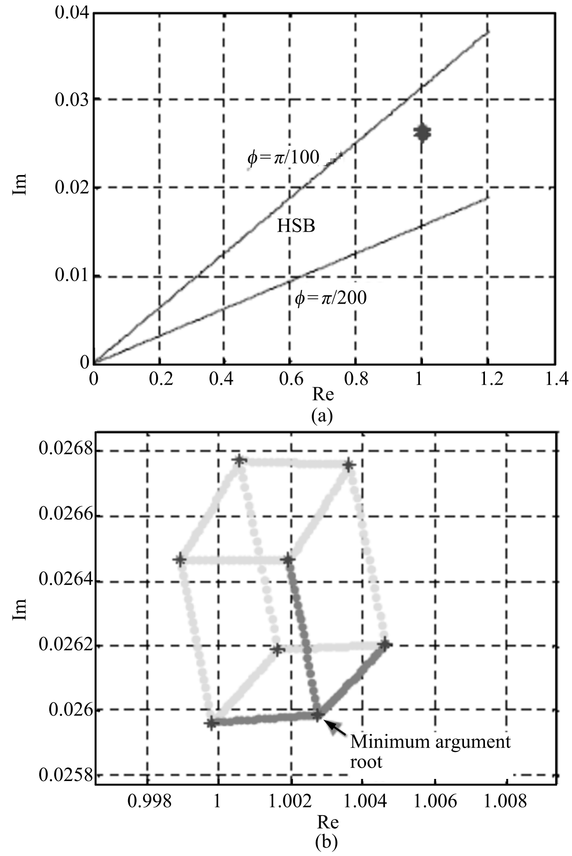

Fig. 8. Root placement for Example 2 ((a) Roots of vertex and edge polynomials; (b) Close view of minimum argument vertex and connected edge polynomials.)

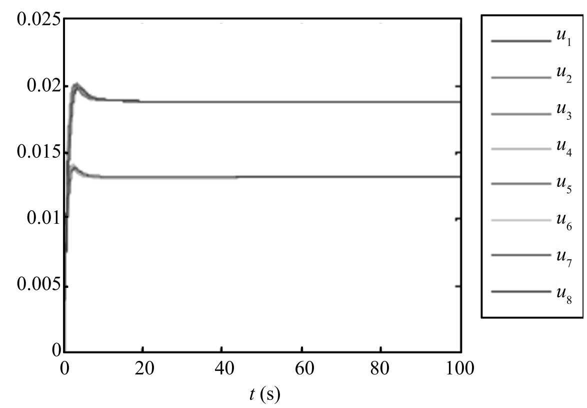

Fig. 9. Step responses of vertex polynomials.

It is convenient to write closed loop transfer function of system:

$$T(s) = \frac{Y(s)}{U(s)} = \frac{12.46s + 64.47}{(39.96 \pm \gamma)s^{1.25} + (12.46 \pm \gamma)s + (65.068 \pm \gamma)}. \tag{24}$$

Then, the characteristic polynomial was obtained as,

$$\Delta(s) = (39.96 \pm \gamma)\ s^{1.25} + (12.46 \pm \gamma)s + (65.068 \pm \gamma). \tag{25}$$

By applying $s = v^{100}$ mapping, the expanded degree integer order characteristic polynomial is written as,

$$\Delta^{100}(v) = (39.96 \pm \gamma)\ v^{125} + (12.46 \pm \gamma) v^{100} + (65.068 \pm \gamma). \tag{26}$$

Vertex polynomials of expanded degree integer order interval characteristic polynomial were obtained as

$$\begin{aligned}
\{\Delta_{u_1} &= \Delta^{100}([(39.96-\gamma)\ (12.46-\gamma)\ (65.068-\gamma)], v),\\
\Delta_{u_2} &= \Delta^{100}([(39.96-\gamma)\ (12.46-\gamma)\ (65.068+\gamma)], v),\\
\Delta_{u_3} &= \Delta^{100}([(39.96-\gamma)\ (12.46+\gamma)\ (65.068-\gamma)], v),\\
\Delta_{u_4} &= \Delta^{100}([(39.96-\gamma)\ (12.46+\gamma)\ (65.068+\gamma)], v),\\
\Delta_{u_5} &= \Delta^{100}([(39.96+\gamma)\ (12.46-\gamma)\ (65.068-\gamma)], v),\\
\Delta_{u_6} &= \Delta^{100}([(39.96+\gamma)\ (12.46-\gamma)\ (65.068+\gamma)], v),\\
\Delta_{u_7} &= \Delta^{100}([(39.96+\gamma)\ (12.46+\gamma)\ (65.068-\gamma)], v),\\
\Delta_{u_8} &= \Delta^{100}([(39.96+\gamma)\ (12.46+\gamma)\ (65.068+\gamma)], v)\}.
\end{aligned}$$

We used 19 polynomials edge sampling. Then, edge polynomials were obtained as,

$$\begin{aligned}
\{\Delta_{e_1} &= \Delta^{100}([(\lambda(39.96-\gamma)\\
&\quad + (1-\lambda)(39.96+\gamma))(12.46-\gamma)(65.068-\gamma)], v),\\
\Delta_{e_2} &= \Delta^{100}([(\lambda(39.96-\gamma)\\
&\quad + (1-\lambda)(39.96+\gamma))(12.46-\gamma)(65.068+\gamma)], v),\\
\Delta_{e_3} &= \Delta^{100}([(\lambda(39.96-\gamma)\\
&\quad + (1-\lambda)(39.96+\gamma))(12.46+\gamma)(65.068-\gamma)], v),\\
\Delta_{e_4} &= \Delta^{100}([(\lambda(39.96-\gamma)\\
&\quad + (1-\lambda)(39.96+\gamma))(12.46+\gamma)(65.068+\gamma)], v),\\
\Delta_{e_5} &= \Delta^{100}([(39.96-\gamma)(\lambda(12.46-\gamma)\\
&\quad + (1-\lambda)(12.46+\gamma))(65.068-\gamma)], v),\\
\Delta_{e_6} &= \Delta^{100}([(39.96-\gamma)(\lambda(12.46-\gamma)\\
&\quad + (1-\lambda)(12.46+\gamma))(65.068+\gamma)], v),\\
\Delta_{e_7} &= \Delta^{100}([(39.96+\gamma)(\lambda(12.46-\gamma)\\
&\quad + (1-\lambda)(12.46+\gamma))(65.068-\gamma)], v),\\
\Delta_{e_8} &= \Delta^{100}([(39.96+\gamma)(\lambda(12.46-\gamma)\\
&\quad + (1-\lambda)(12.46+\gamma))(65.068+\gamma)], v),\\
\Delta_{e_9} &= \Delta^{100}([(39.96-\gamma)(12.46-\gamma)(\lambda(65.068-\gamma)\\
&\quad + (1-\lambda)(65.068+\gamma)\ )], v),\\
\Delta_{e_{10}} &= \Delta^{100}([(39.96-\gamma)(12.46+\gamma)(\lambda(65.068-\gamma)\\
&\quad + (1-\lambda)(65.068+\gamma)\ )], v),\\
\Delta_{e_{11}} &= \Delta^{100}([[(39.96+\gamma)(12.46-\gamma)(\lambda(65.068-\gamma)\\
&\quad + (1-\lambda)(65.068+\gamma)\ )], v),\\
\Delta_{e_{12}} &= \Delta^{100}([(39.96+\gamma)(12.46+\gamma)(\lambda(65.068-\gamma)\\
&\quad + (1-\lambda)(65.068+\gamma)\ )], v)\}.
\end{aligned}$$

Figs. 10 (a)-10 (h) show roots of vertex and edge polynomials in the first Riemann sheet for various values of γ. Table II lists minimum argument of vertex roots with respect to value of γ. Graphical results shown in Fig.10 indicate that interval uncertain control system is robust stable for $\gamma \leq 27$.

In this example, all edge and vertex polynomials (AEAV) method[14] requires the test of 236 polynomials ($19.3.2^2+2^3$), minimum argument vertex with connected edge polynomials (MVCE) method requires the test of 65 polynomials ($19.3+2^3$) and minimum argument vertex polynomials (MV) method requires the test of 8 polynomials (2^3). Examples numerically reveal that MVCE and MV can reduce computational complexity of robust stability analysis based on edge theorem; however there is need for theoretical verification of basic assumptions of MVCE and MV approaches.

VI. Conclusions

This study confirms that Hurwitz stability analysis of fractional order characteristic polynomials is valid in v plane under power mapping. As known, it is difficult to calculate root locus of fractional order characteristic polynomials in s domain. The $s = v^m$ power mapping significantly simplifies stability analyses of fractional order polynomials. The problem turns into the Hurwitz stability analysis of expanded degree integer order polynomials in the first Riemann sheet.

It is important to investigate impacts of power mapping on root locus and stability related properties. Preliminarily, this paper revealed properties of $s = v^m$ mapping related with stability analysis and root locus: It was shown that root argument and magnitude relations are conserved under power mapping. This is an important remark of power mapping that leads to conformity of root locus analysis given in v plane for the fractional order systems defined in s plane. The conservation of argument and magnitude relations leads to conservation of the geometrical properties of root constellation under power mapping transformations between complex s and v planes. Thus, the minimum argument root of expanded degree integer order polynomials in complex v plane is also

TABLE II
MINIMUM ARGUMENT OF VERTEX POLYNOMIAL ROOTS AND SYSTEM STABILITY FOR VARIOUS γ

Value of γ	Minimum angle vertex polynomial number (1-8)	Minimum argument of vertex polynomials (Radian)	Robust stability
1	6	0.0263	Stable
5	6	0.0258	Stable
10	6	0.0254	Stable
15	1	0.0246	Stable
20	1	0.0231	Stable
25	1	0.0193	Stable
27	1	0.0158	Stable
28	1	0.0128	Unstable

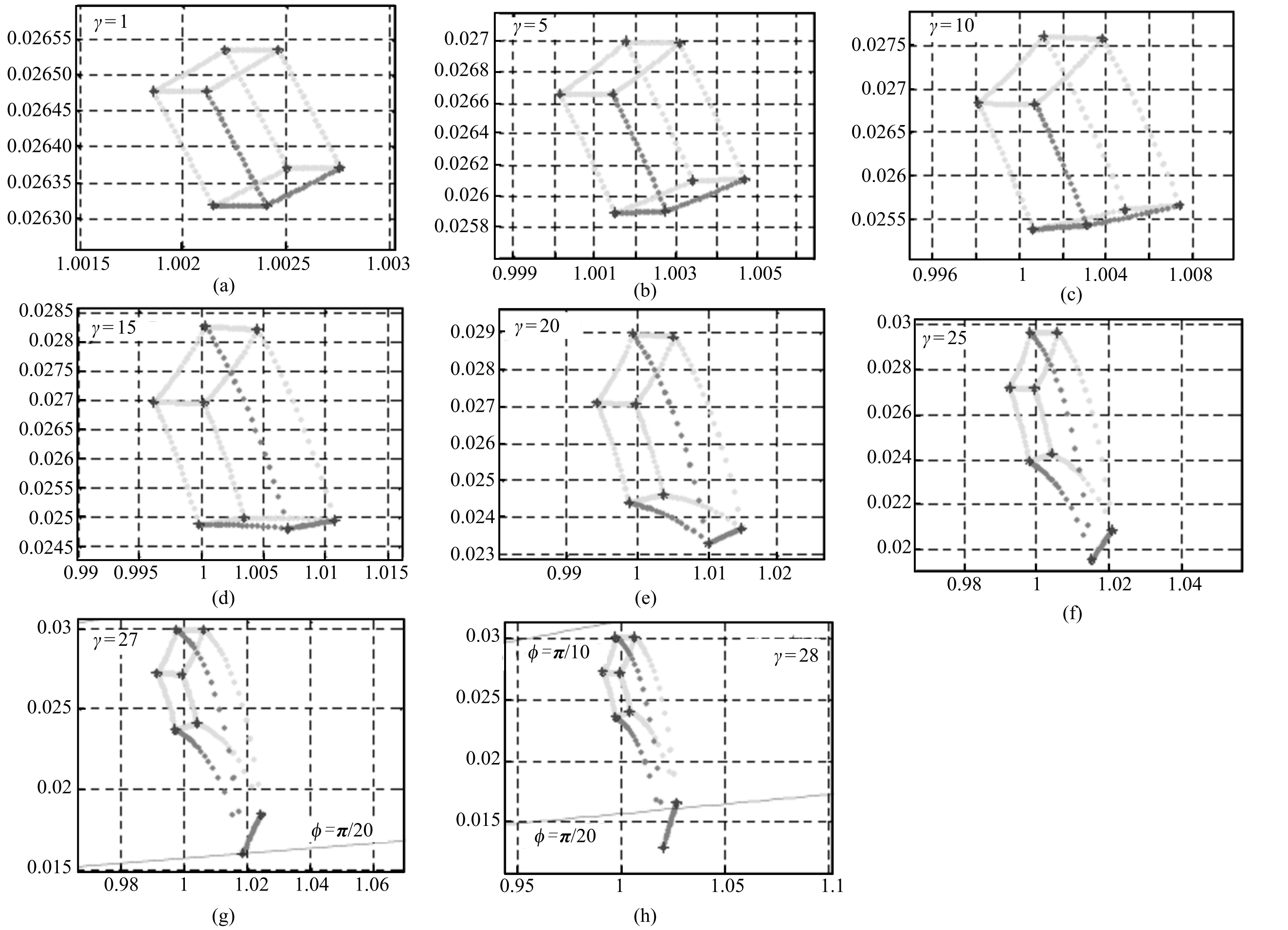

Fig. 10. Root regions of vertex and edge polynomials in the first Riemann sheet for (a) $\gamma = 1$, (b) $\gamma = 5$, (c) $\gamma = 10$, (d) $\gamma = 15$, (e) $\gamma = 20$, (f) $\gamma = 25$, (g) $\gamma = 27$ and (h) $\gamma = 28$.

minimum argument root of fractional order polynomial in the s plane. This property provides the validity of Hurwitz stability and implication of edge theorem under power mapping and it makes possible the robust stability analysis of fractional order interval polynomials according to robust stability of expanded degree integer order polynomials complex v plane.

To utilize edge theorem based approach for robust stability analysis of fractional order control systems. Author numerically demonstrated two robust stability analysis approaches based on minimum root argument analyses of vertex and exposed edge polynomials in v plane. These approaches were shown to reduce the number of test polynomials for the parametric robust stability analyses of fractional order systems. It was observed in numerical calculations that the test of only vertex polynomials can significantly reduce computational complexity of robust stability analyses for interval characteristic polynomials with positive real coefficients. It should be noted that results are valid under the assumption that minimum argument root comes from vertex and/or connected edge polynomials of hyper-rectangle. Results of numerical examples confirm the validity of this assumption. However, there is need for a future study addressing properties of zeros in polynomial arithmetics for the theoretical proof of this assumption.

This study contributes to advance our understanding on implications of power mapping for root locus and stability properties of fractional order systems.

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